Paradoxical discussions on sediment transport formulas

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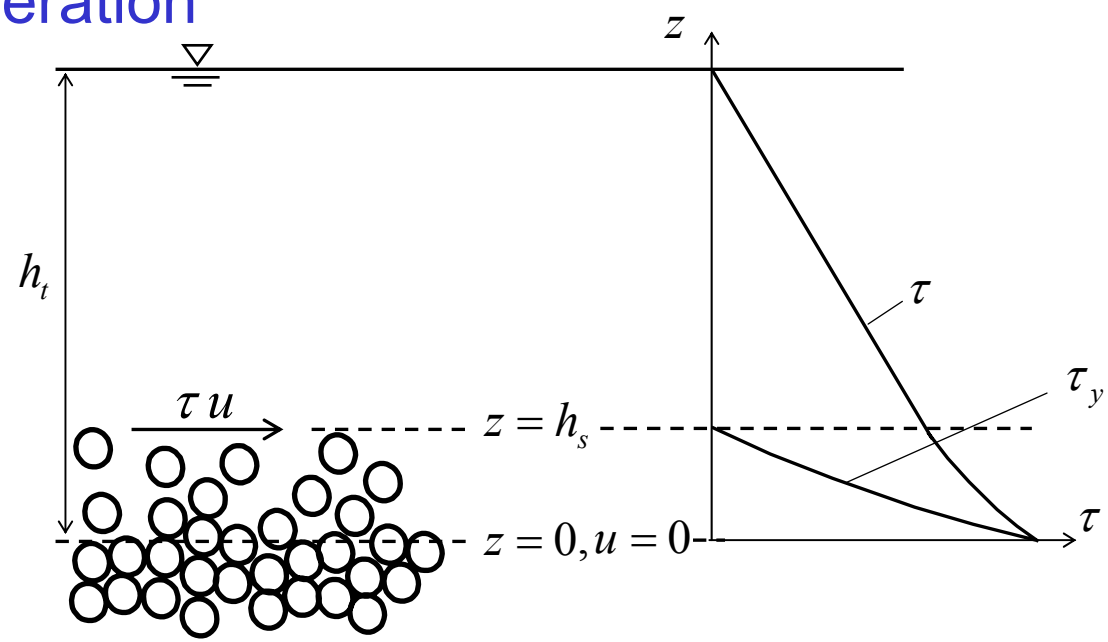
Bed-load formulas characterized by a form, $\tau_*^{3/2}$

We have employed a bed load formula which predicts the transport rate proportional to $\tau_*^{3/2}$. Although no serious problems might be caused practically, some important matters should be studied in view of sediment mechanics.

A new formula derived from a view of continuum mechanics

We will obtain a different formula using a velocity profile and thickness of bed-load layer which are evaluated in terms of governing equations with constitutive relations for water-sediment mixture.

Interpretation of bed-load formulas derived by energetic consideration



$$q_b = \int_0^{h_s} c\,u\,dz \cong \bar{c}_s h_s u_s$$

$q_b \sim \tau u \sim u_*^3$ (u_* : shear velocity)

In these, control volume of bed-load layer is not defined.

$\bar{c}_s h_s$

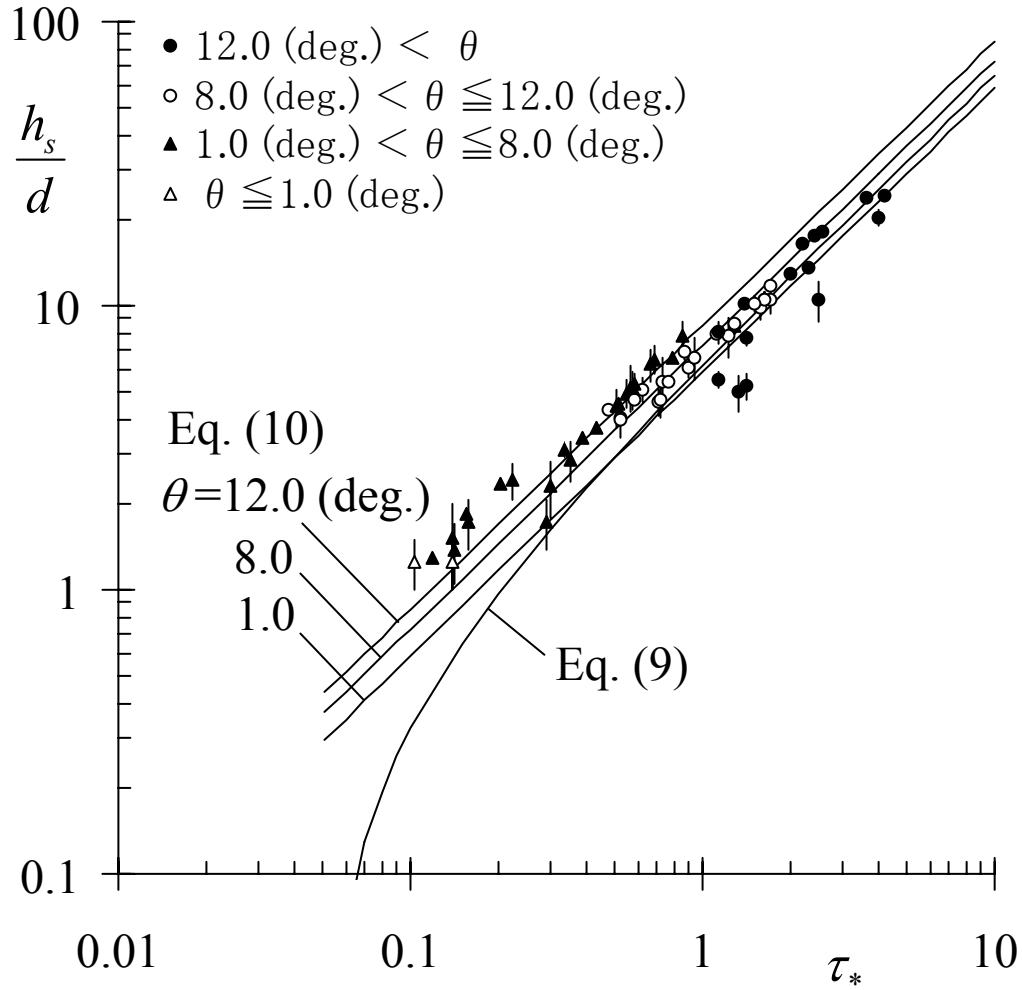
$$\bar{c}_s h_s = \frac{\tau_b}{(\sigma-\rho)g}\frac{1}{\mu}$$

$$\frac{h_s}{d} = \frac{1}{\bar{c}_s \mu}\tau_*$$

τ_b : Bed shear stress

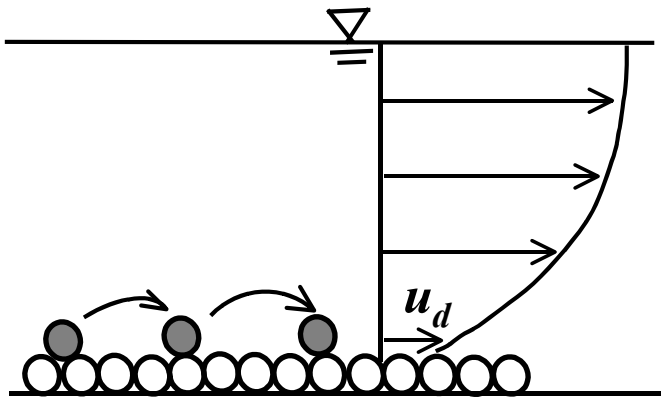
μ : Friction coefficient

$$\tau_* = \frac{u_*^2}{(\sigma/\rho - 1)gd}$$



Relationship between non-dimensional bed shear stress and thickness of bed-load layer

u_s (Velocity of sediment particle)



$$(m+m')\frac{du_s}{dt} = \frac{1}{2}\rho C_D |u_d - u_s|(u_d - u_s)A_s - \mu N'$$

$$u_s \sim u_d \sim u_* \text{ (or } u_* - u_{*c})$$

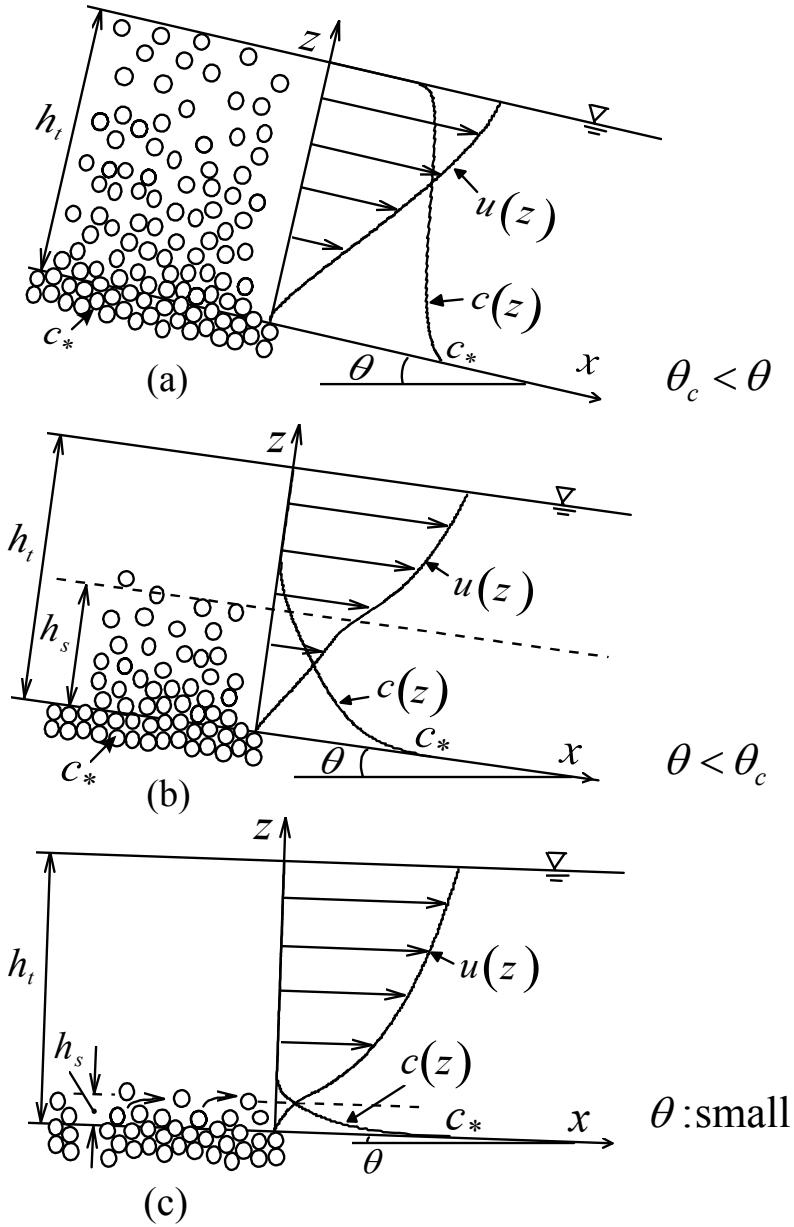
Combining u_s and $\bar{c}_s h_s$,

$$q_b \sim u_*^3 \text{ or } q_{b*} \sim \tau_*^{3/2}$$

$$q_{b*} = \frac{q_b}{\sqrt{(\sigma/\rho - 1)gd^3}}$$

Sediment transport mode

A new formula derived from continuum mechanics



Change of bed-load layer thickness

$$q_b = \int_0^{h_s} c\,u\,dz \cong \bar{c}_s h_s u_s$$

$\bar{c}_s$: Sediment concentration of bed-load layer

h_s : Thickness of bed-load layer

u_s : Average velocity of bed-load layer

Relation of energy dissipation and stress

Mass conservation law

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m u_i}{\partial x_i} = 0 \qquad (1)$$

Momentum conservation law

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = b_i - \frac{1}{\rho_m}\frac{\partial p}{\partial x_j}\delta_{ij} + \frac{1}{\rho_m}\frac{\partial \tau_{ij}}{\partial x_j} \qquad (2)$$

Energy conservation law

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \rho_m b_i u_i - \frac{\partial p u_i}{\partial x_j}\delta_{ij} + \frac{\partial \tau_{ij} u_i}{\partial x_j} - \Phi \qquad (3)$$

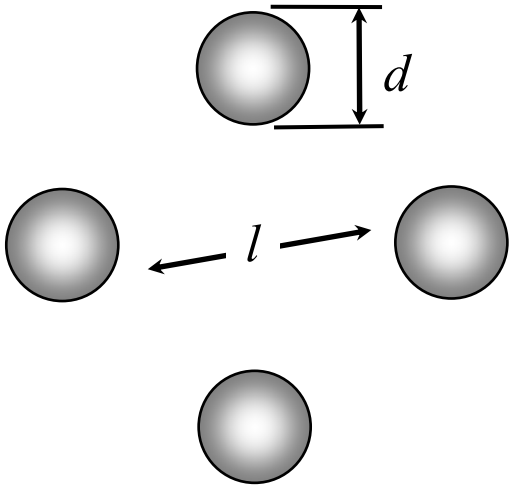
$$\rho_m = \sigma c + (1-c)\rho \qquad k = \frac{1}{2}\rho_m u_i u_i$$

Incompressibility → $\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j}$ (4)

ρ_m : mass density of sediment mixture
u_i : velocity
b_i : body force
τ_{ij} : shear stress tensor
p : isotropic component of stress
k : kinetic energy
Φ : energy dissipation rate
σ : mass density of sediment particles
ρ : mass density of water
c : sediment concentration by volume in the mixture

Parameters associated with energy dissipation

Energy dissipation	Parameters
Particle to particle contacts	Inter-particle friction angle ϕ_s
Inelastic particle to particle collisions	# Restitution coefficient e # Mass density of sediment particles σ # Sediment size d # Sediment concentration c
Shearing of interstitial water	# Kinematic viscosity of water ν # Mass density of water ρ # Scale of pore water region l



$$l \propto (1-c)^{1/3}$$

Scale of pore water region

Relations for shear stress and pressure in uniform flows

$$\tau = \tau_y + \tau_f + \tau_d \tag{5}$$

$$p = p_s + p_w + p_d \tag{6}$$

$$\tau_y = p_s \tan\phi_s \tag{7}$$

$$\tau_f = \rho\nu_f(\partial u/\partial z) + \rho k_f d^2 \frac{(1-c)^{5/3}}{c^{2/3}}(\partial u/\partial z)^2 \tag{8}$$

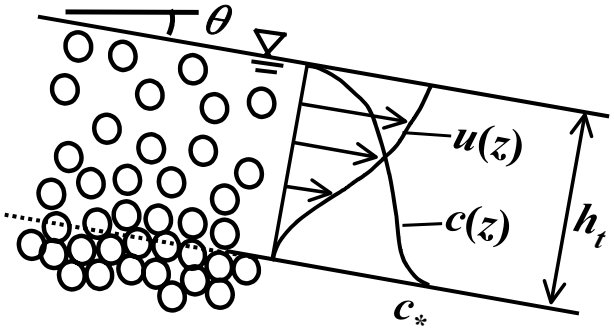
$$\tau_d = k_d(1-e^2)\sigma d^2 c^{1/3}(\partial u/\partial z)^2 \tag{9}$$

$$p_d = k_d \sigma e^2 d^2 c^{1/3}(\partial u/\partial z)^2 \tag{10}$$

$$p_s/(p_s + p_d) \equiv f(c) = (c/c_*)^{1/n} \tag{11}$$

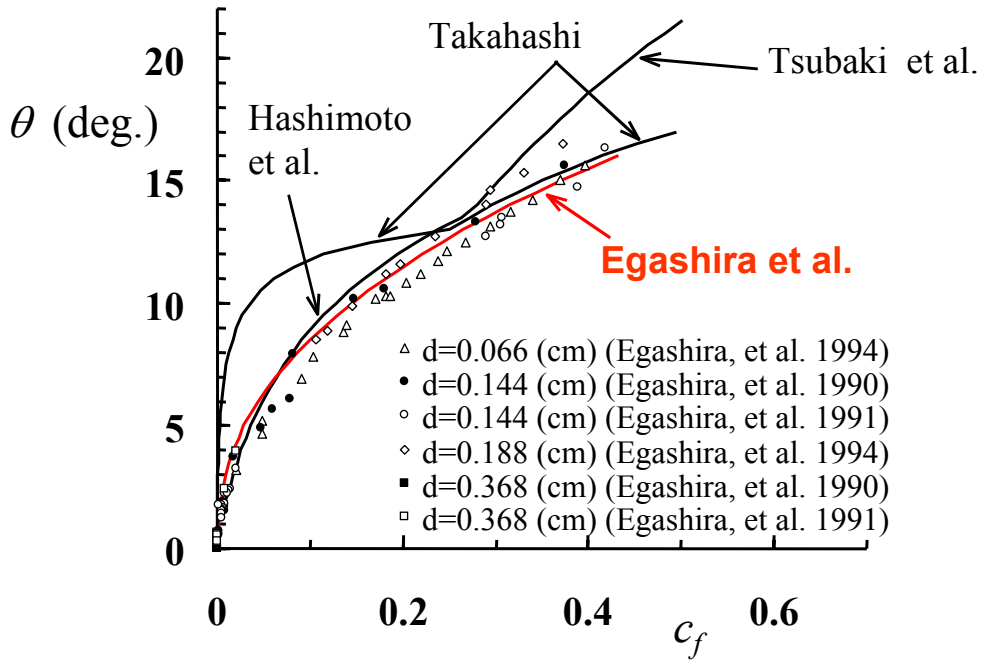
in which τ_y : yield stress
τ_f : shear stress supported by interstitial water
ν_f : kinematic viscosity of the liquid phase
τ_d : shear stress due to inelastic particle to particle collisions
p_s : pressure of static interparticle contacts
p_d : dynamic pressure due to inelastic particle collisions
k_d= 0.0828 and k_f= 0.16 are empirical constants.
n : an empirical constant (n= 5.0)

Flux sediment concentration calculated using constitutive relations for debris flows



$$c_f = \int_0^{h_t} cudz \Big/ \int_0^{h_t} udz \qquad ※\ c_f \neq \bar{c}$$

$$\bar{c} = \int_0^{h_t} cdz \Big/ h_t$$



Relation between flux sediment concentration and equilibrium bed slope

Derivation of a bed-load formula

(1) Exact solution

$$q_b = \int_0^{h_s} u\,c\,dz$$

$$q_{b*} \equiv \frac{q_b}{\sqrt{(\sigma/\rho - 1)gd^3}}$$

(2) Approximate solution

$$q_b = \int_0^{h_s} u\,c\,dz = \bar{c}_s h_s \times u_s$$

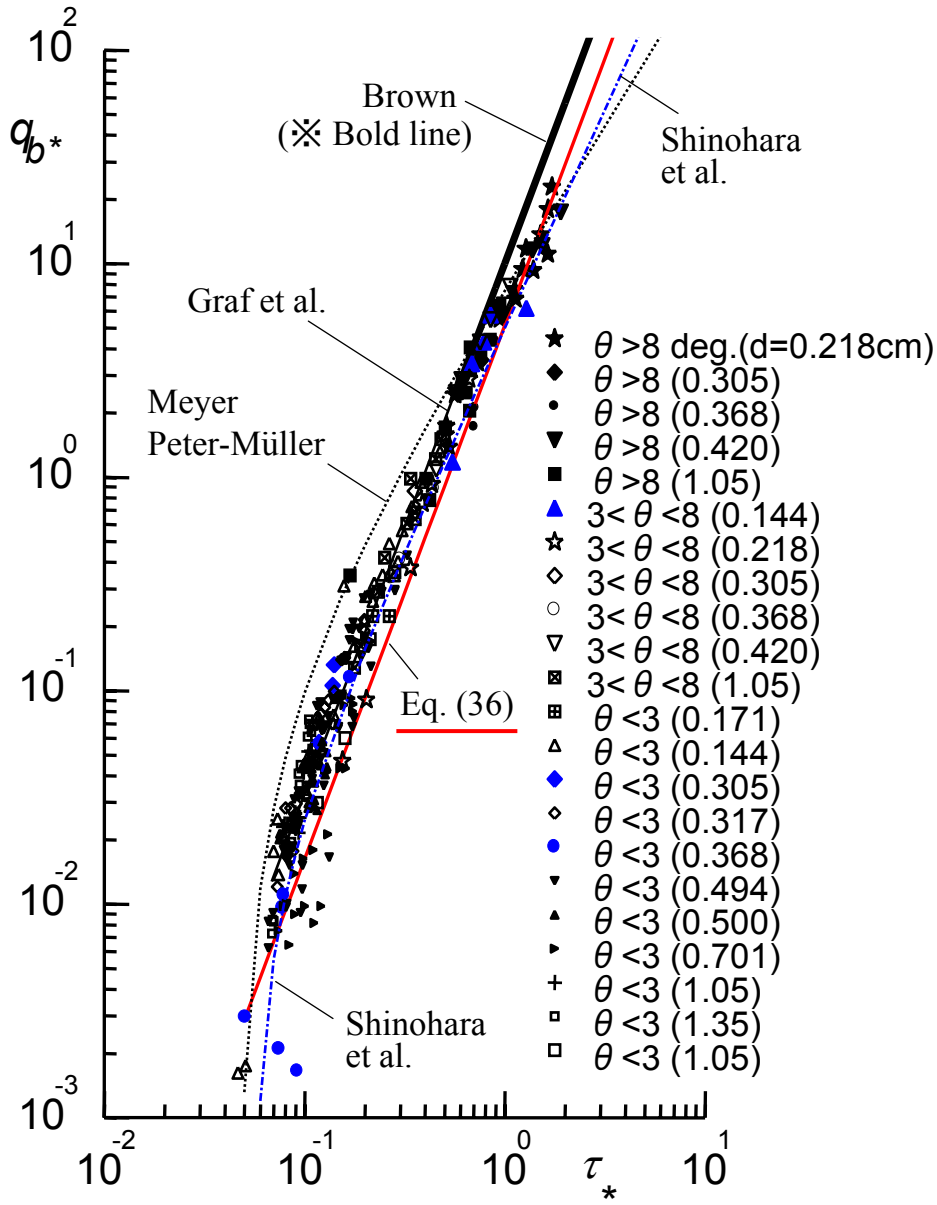
$$\bar{c}_s = const.$$

$$\frac{u_s}{u_*} = \frac{4}{15}\frac{K_1 K_2}{\sqrt{f_f + f_d}}\tau_*$$

$$K_1 = \frac{1}{\cos\theta}\frac{1}{\tan\phi_s - \tan\theta} \qquad K_2 = \frac{1}{\bar{c}_s}\left[1 - \frac{\tan\theta}{(\sigma/\rho - 1)\bar{c}_s(\tan\phi_s - \tan\theta)}\right]^{1/2}$$

$$f_f = k_f \frac{(1-\bar{c}_s)^{5/3}}{\bar{c}_s^{\,2/3}} \qquad f_d = k_d\left(1 - e^2\right)(\sigma/\rho)\bar{c}_s^{-1/3}$$

$$q_{b*} = \frac{4}{15}\frac{K_1^{\,2} K_2}{\sqrt{f_f + f_d}}\tau_*^{\,5/2}$$



Relation of bed-load rate and bed shear stress

Concluding remarks

(1) Some problems which should be studied in the bed-load formulas characterized by $\tau_*^{3/2}$ are extracted; unclear boundaries among stationary sediment layer, bed-load layer and clear water region and mechanics of bed load layer and associated problems.

(2) A bed load formula is proposed, which is derived using governing equations and constitutive relations developed for debris flow and has a form very different from $\tau_*^{3/2}$. The equation can predict a lower edge of flume data in wide range of bed shear stress, although no data-fittings are performed.

There are problems to be studied, such as constitutive equations of bed-load layer.

We will need governing equations written in mass-, momentum- and energy- conservation base for the bed-load layer in order to solve complex sediment problems.