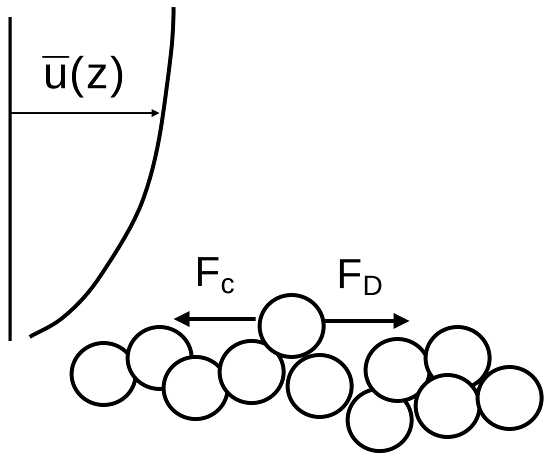
THRESHOLD OF MOTION

Brief sketch of the concept of critical Shields stress

- μ_c = coefficient of Coulomb friction, θ_r = angle of repose of sediment
- u_f = effective flow velocity acting on an “exposed” bed particle
- c_D = drag coefficient for an “exposed bed particle

$$\theta_r = \tan^{-1}(\mu_c) = 30° \text{ to } 40°$$



$$F_D = \frac{1}{2}\rho\pi c_D\left(\frac{D}{2}\right)^2 u_f^2$$ Impelling fluid drag force

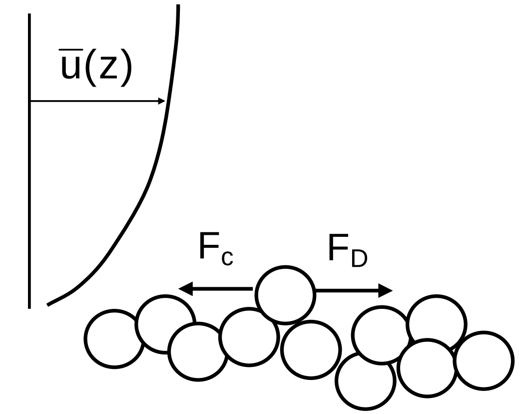
$$F_g = \frac{4}{3}\rho\pi Rg\left(\frac{D}{2}\right)^3$$ Submerged weight of grain

$$F_c = \mu F_g$$ Coulomb resistive force

Threshold of motion: $F_D = F_c$ or thus

$$\frac{u_f^2}{RgD} = \frac{4}{3}\frac{\mu_c}{c_D}$$

SHIELDS DIAGRAM



$$\frac{u_f^2}{RgD} = \frac{4}{3}\frac{\mu_c}{c_D}$$

Now let $u_f = \bar{u}\big|_{z=n_f D}$ where n_f is an order-one constant

In addition, let $k_s = n_k D$ where n_k is another order-one constant

Thus $$\frac{u_f}{u_*} = \frac{1}{\kappa}\ell n\left(30\frac{z}{k_s}\right)\Bigg|_{z=n_f D} = \frac{1}{\kappa}\ell n\left(30\frac{n_f}{n_k}\right) \equiv F_u$$

Reduce to get Shields criterion

Define the dimensionless Shields number τ^* characterizing sediment mobility as

$$\tau^* \equiv \frac{\tau_b}{\rho RgD} = \frac{u_*^2}{RgD}$$

Then the critical Shields number τ_c^* at the threshold of motion is given as (e.g. Ikeda, 1982)

$$\tau_c^* = \frac{4}{3}\frac{\mu_c}{c_D F_u^2} \sim 0.03$$

SHIELDS DIAGRAM contd.

The previous analysis is merely a sketch.
In order to include the full range of grain sizes it is necessary to include viscous effects (e.g. Ikeda, 1982).

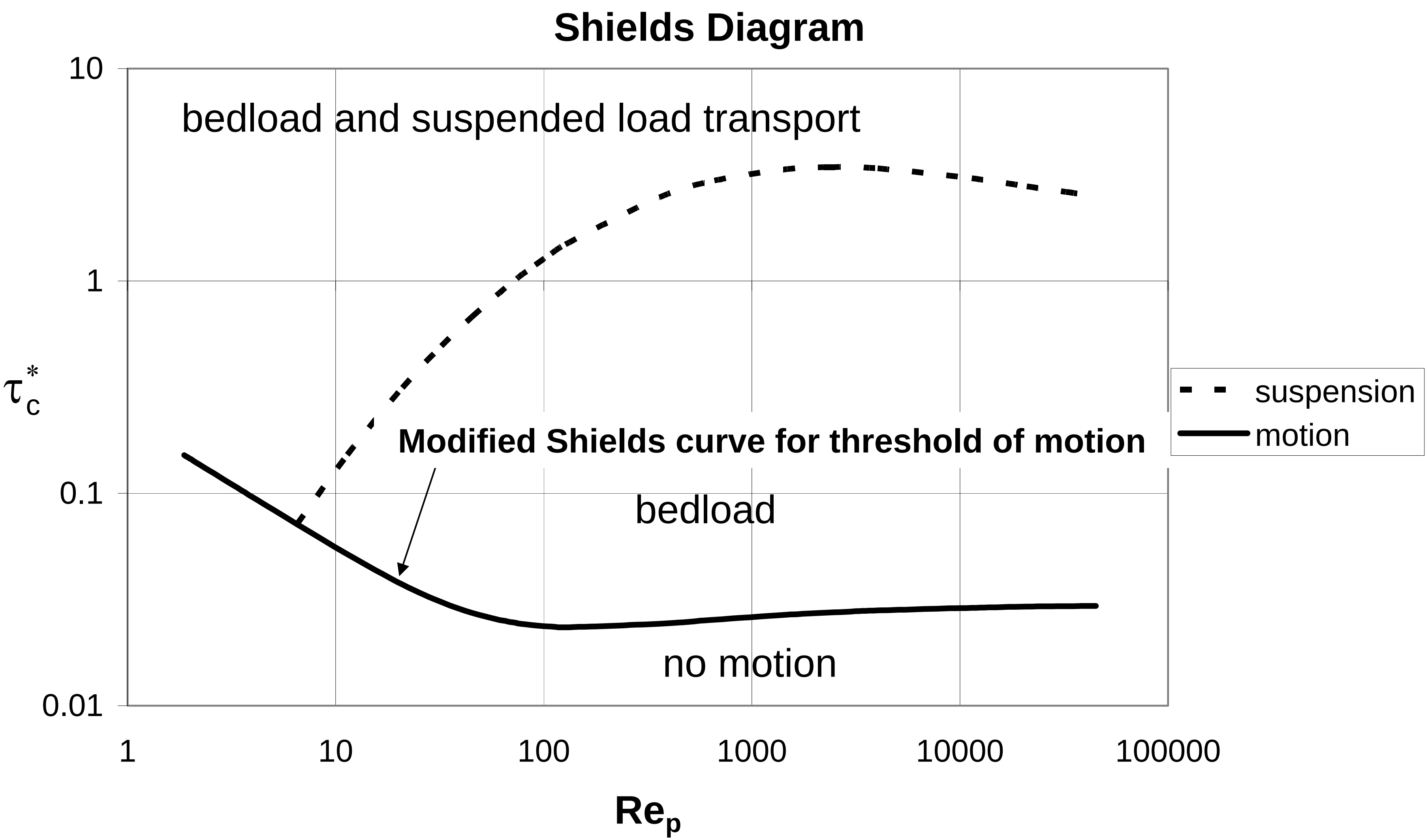
Parker's emendation of Brownlie's (1982) fit to the Shields (1936) curve so as to agree with results of Neill (1968):

$$\tau_c^* = 0.5 * [0.22\,\mathbf{Re}_p^{-0.6} + 0.06 \cdot 10^{(-7.7\,\mathbf{Re}_p^{-0.6})}]$$

$$\mathbf{Re}_p = \frac{\sqrt{RgD}\,D}{\nu}$$

Note that $\tau_c^* \to 0.03$ as $\mathbf{Re}_p \to \infty$

National Center for Earth-surface Dynamics:
Renesse 2003: Non-cohesive Sediment Transport



CASE OF SIGNIFICANT STREAMWISE SLOPE

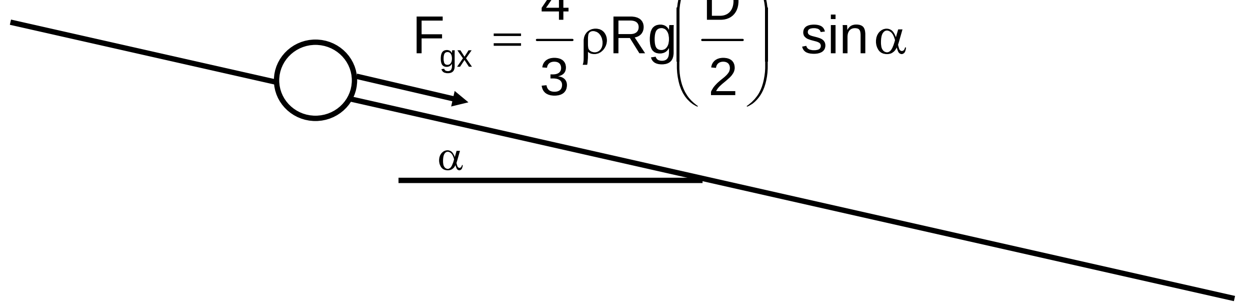
Let α denote the angle of streamwise tilt of the bed, so that

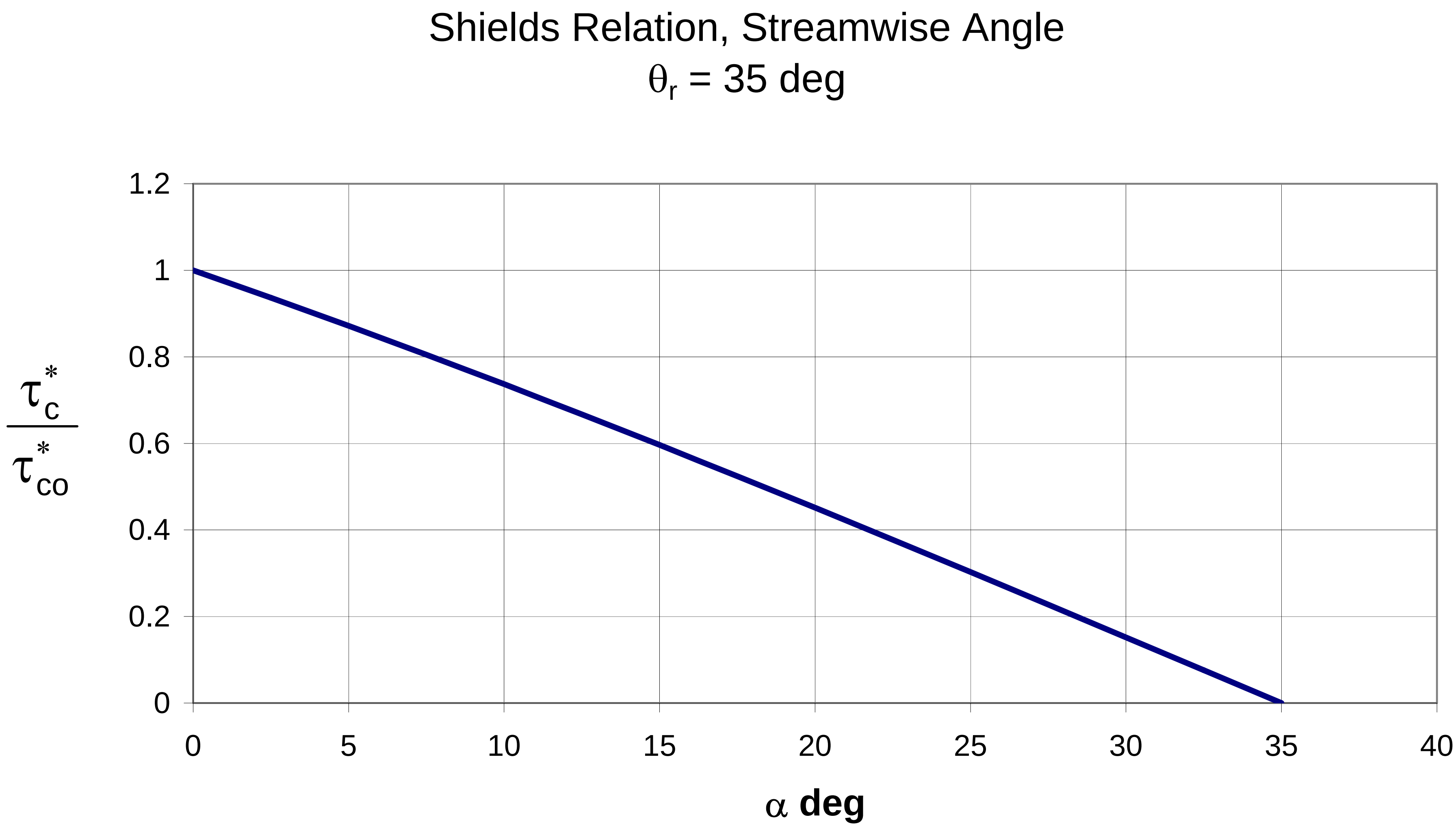
$$S = \tan\alpha$$

Let the value of τ_c^* predicted by the previous formula be amended to τ_{co}^*, where τ_{co}^* denotes the critical Shields stress for the case S << 1 (most cases of interest). In the case of significant streamwise tilt,

$$\tau_c^* = \tau_{co}^* \cos\alpha\left(1 - \frac{\tan\alpha}{\mu_c}\right)$$

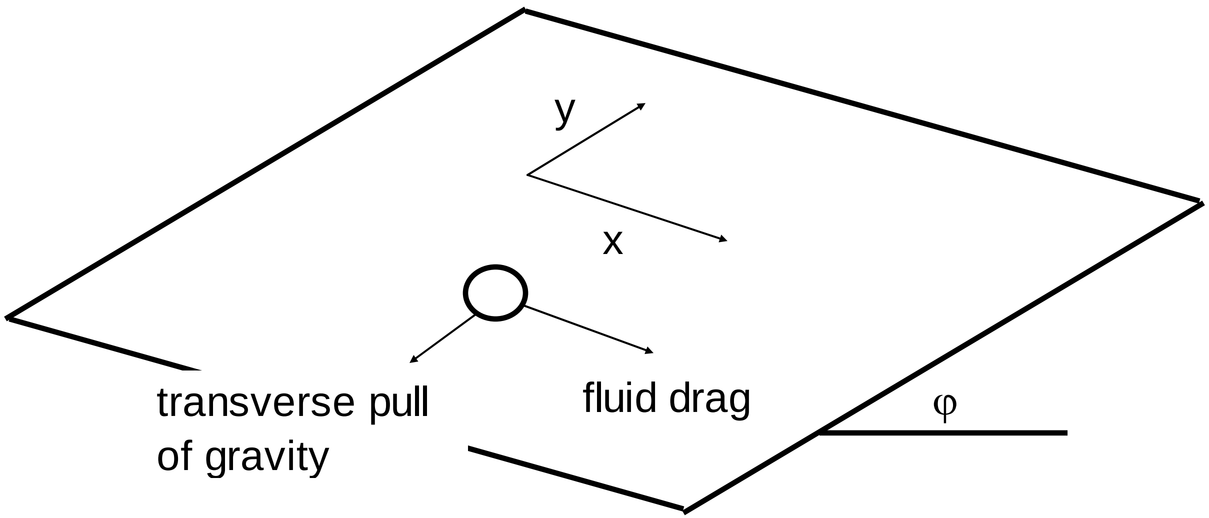
$$F_{gx} = \frac{4}{3}\rho R g \left(\frac{D}{2}\right)^3 \sin\alpha$$




Shields Relation, Streamwise Angle
θr = 35 deg
τc* / τco*
0
0.2
0.4
0.6
0.8
1
1.2
0
5
10
15
20
25
30
35
40
α deg

CASE OF TRANSVERSE SLOPE

Let φ denote the angle of transverse tilt of the bed



$$\tau_c^* = \tau_{co}^* \cos\varphi \left(1 - \frac{\tan^2\varphi}{\mu_c^2}\right)^{1/2}$$

A general formulation of the threshold of motion for arbitrary bed slope is given in Seminara et al. (2002).

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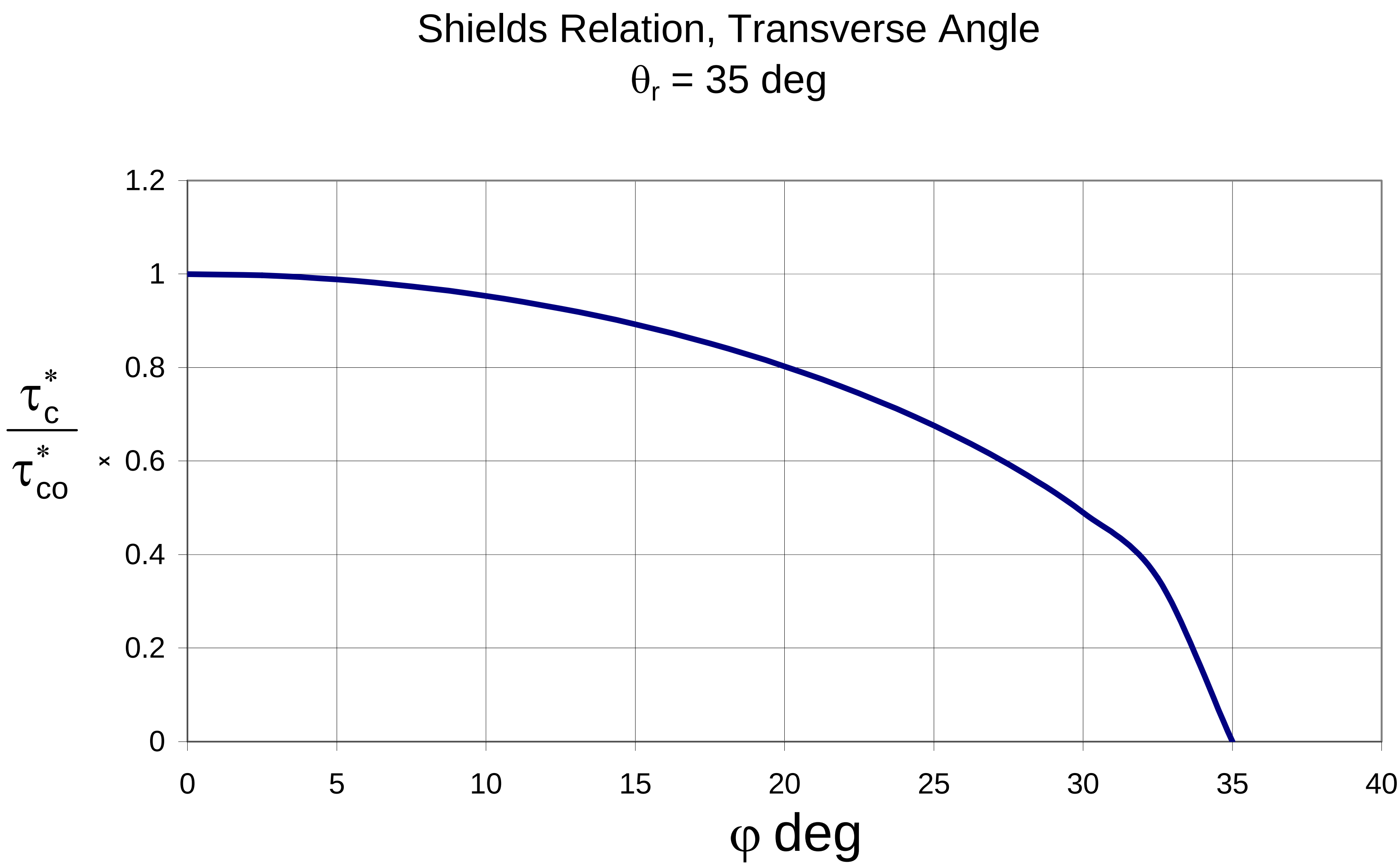

CONDITION FOR "SIGNIFICANT" SUSPENSION OF SEDIMENT

Typical near-bed velocity fluctuation associated with turbulence ~ fall velocity, or thus

$$u_* = v_s$$

Reducing with fall velocity relation,

$$\frac{u_*}{\sqrt{RgD}} = \sqrt{\tau^*} = \frac{v_s}{\sqrt{RgD}} = \mathbf{R}_f(\mathbf{Re}_p)$$

Denote threshold Shields stress for significant suspension as $\tau_{sus}{}^*$

$$\tau^*_{sus} = \tau^*_{sus}(\mathbf{Re}_p)$$

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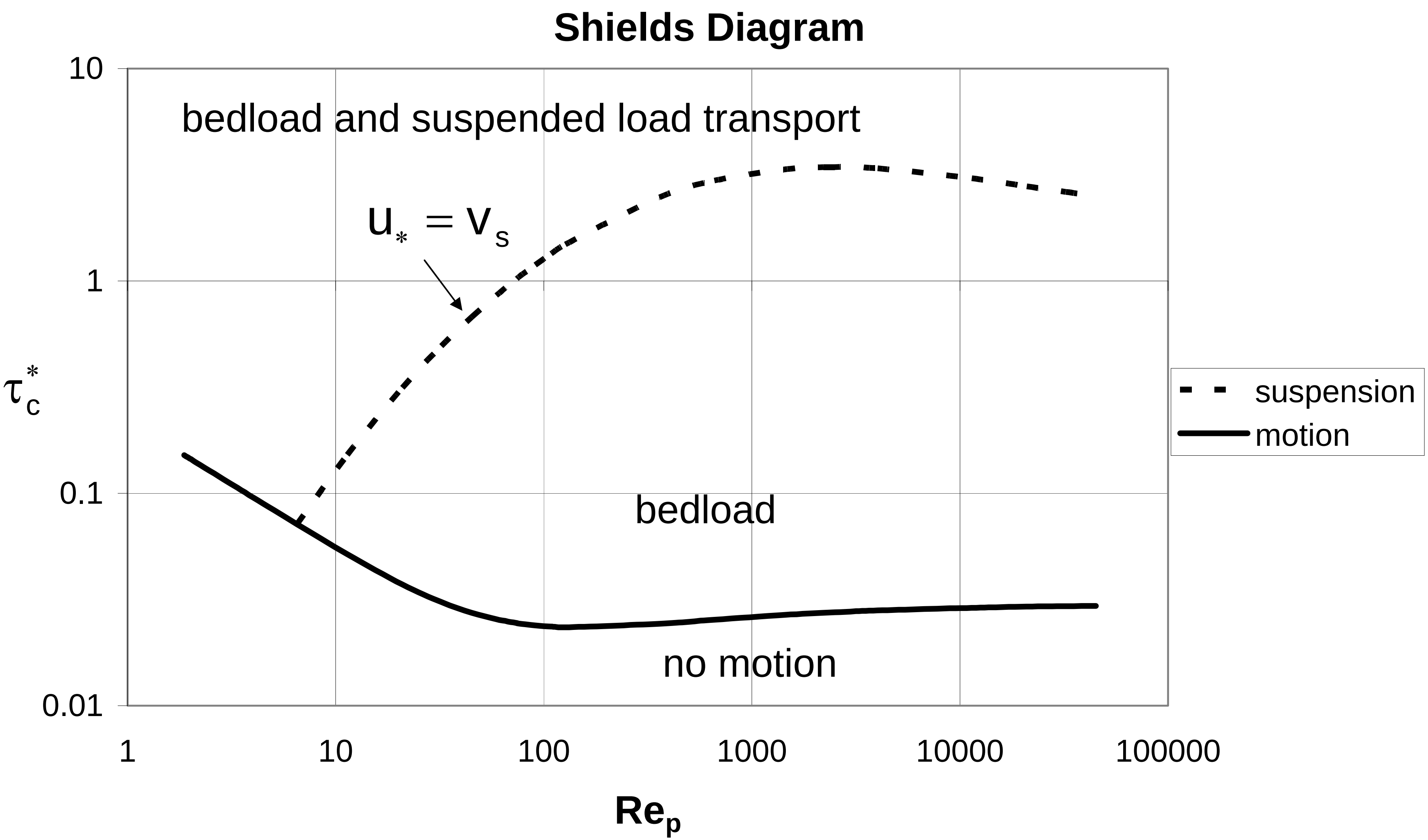
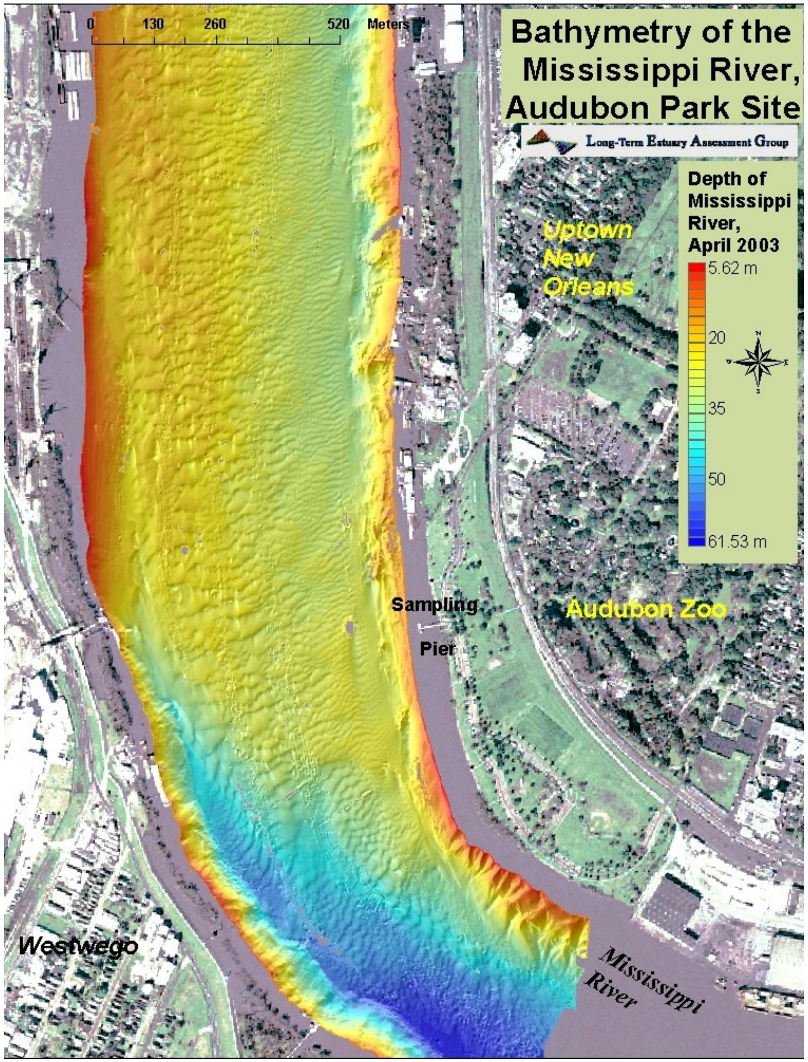


SKIN FRICTION AND FORM DRAG

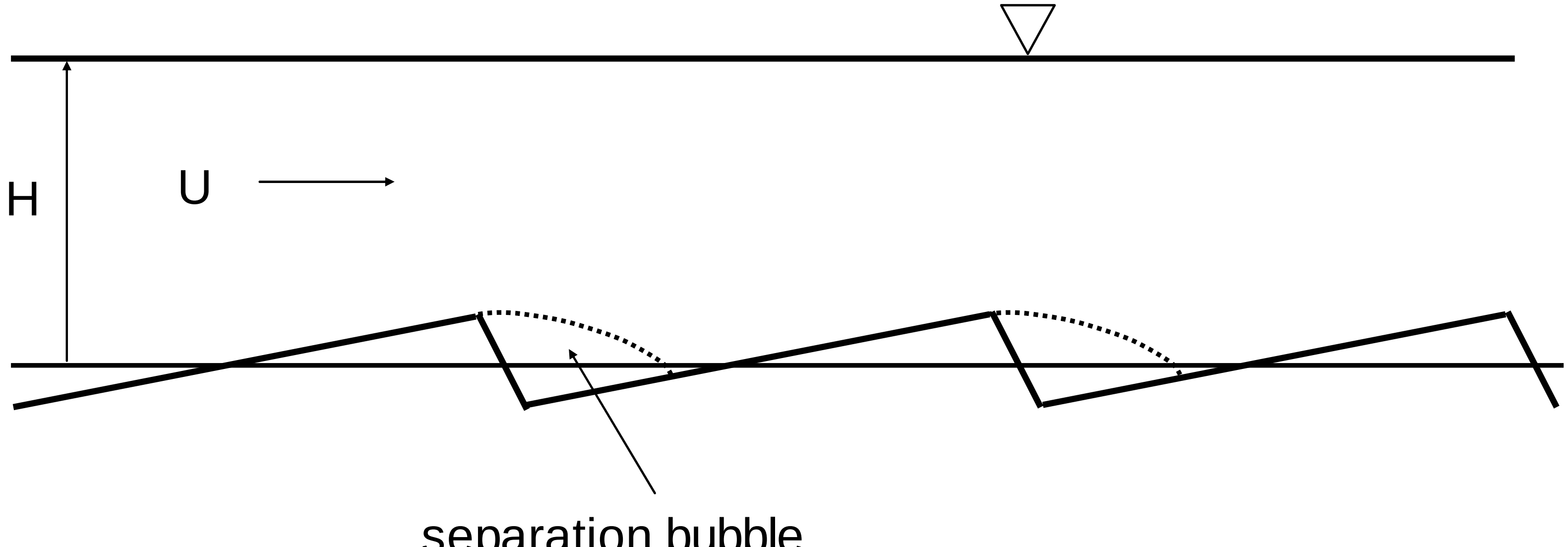


Sediment transport often creates bedforms such as dunes, which in turn affect sediment transport

EINSTEIN DECOMPOSITION

Einstein (1950); Einstein and Barbarossa (1952)

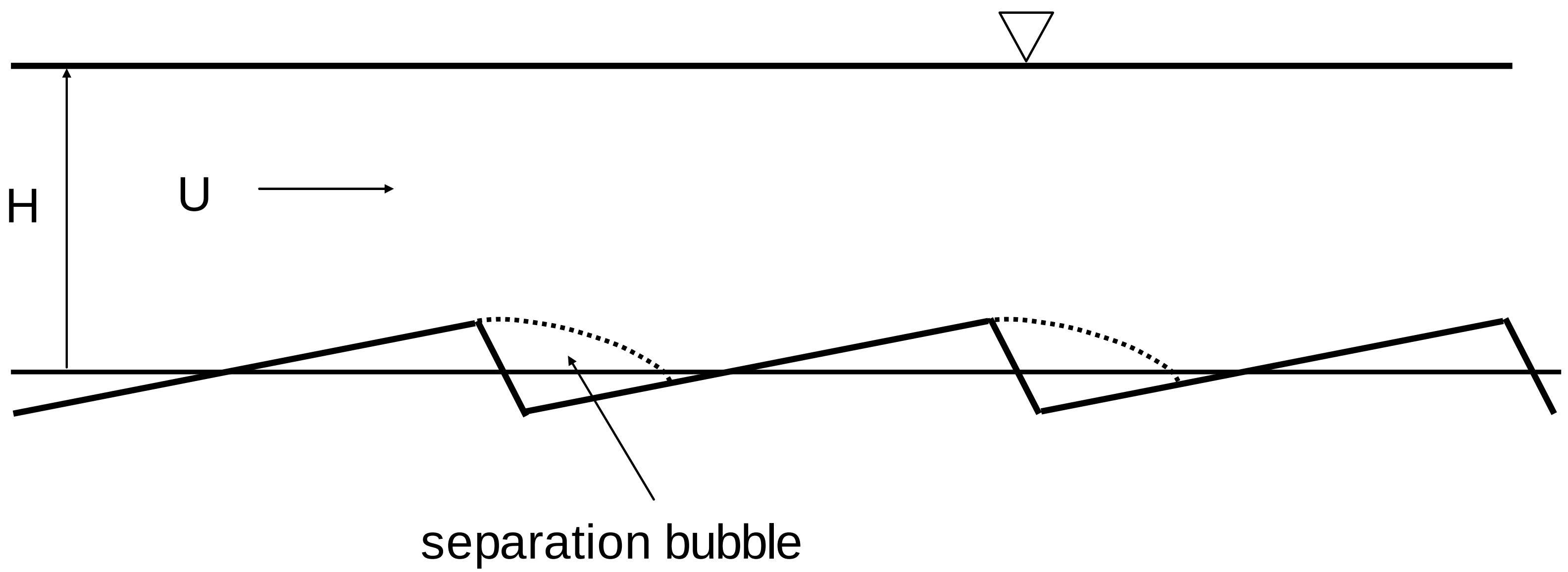
When bedforms are present, much of the resistance to the flow is associated with energy loss in the lee of the bedforms. This “form drag” cannot be expected to contribute to bedload transport or entrainment into suspension. Only the “skin friction” should mobilize sediment



EINSTEIN DECOMPOSITION contd.

Consider an equilibrium flow over a bed with mean streamwise slope S that is covered with bedforms.
The flow has average depth H and velocity U averaged over depth the bedforms. The boundary shear stress averaged over the bedforms is given by the relation

$$\tau_b = \rho C_f U^2 = \rho g H S$$

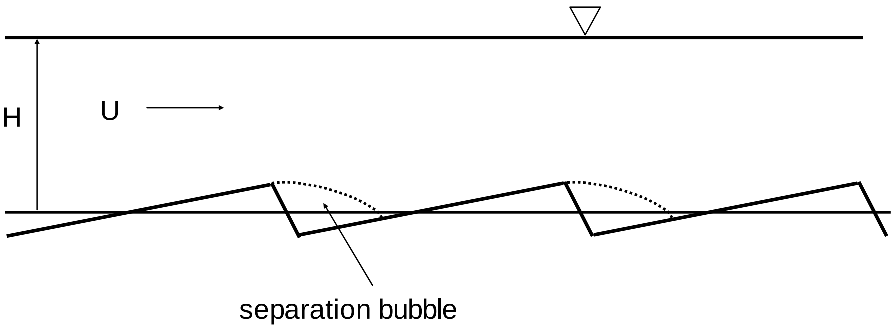


EINSTEIN DECOMPOSITION contd.

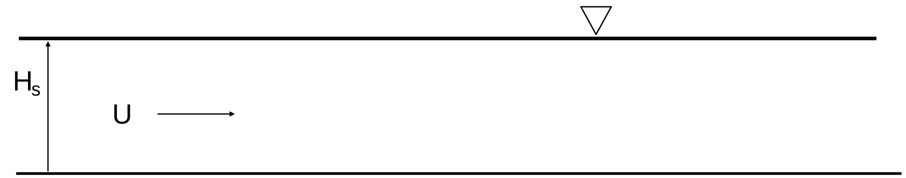
Now smooth out the bedforms, “glue” the sediment to the bed so it remains flat but offers the same microscopic roughness as the case with dunes, and run a flow over it with the same mean velocity U and bed slope S. In the absence of the bedforms, the resistance is skin friction only. Due to the absence of bedforms the skin friction coefficient C_{fs} and the flow depth H_s should be less than the corresponding values with bedforms.

$$\tau_{bs} = \rho C_{fs} U^2 = \rho g H_s S$$

Skin friction + form drag



Skin friction only



The difference between the two characterizes form drag.

EINSTEIN DECOMPOSITION contd.

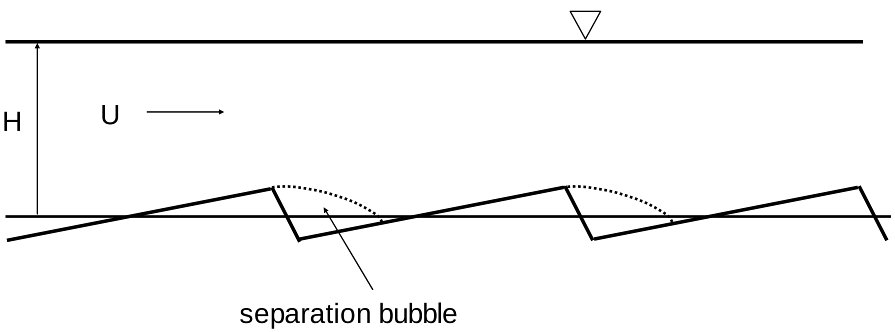
$\tau_{bf} = \tau_b - \tau_{bs}$ = mean bed shear stress due to form drag of bedforms
$C_{ff} = C_f - C_{fs}$ = friction coefficient associated with form drag
$H_f = H - H_s$ = mean depth associated with form drag

$$\tau_{bs} = \rho C_{fs} U^2 = \rho g H_s S \qquad \tau_{bf} = \rho C_{ff} U^2 = \rho g H_f S$$

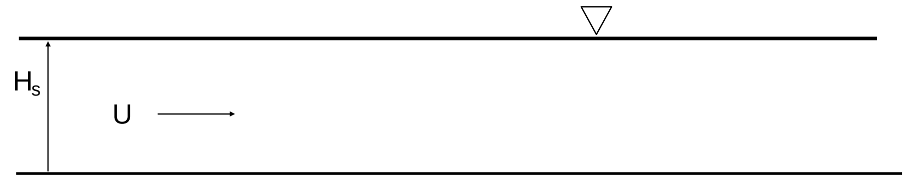
$$\tau_{bs} + \tau_{bs} = \rho\left(C_{fs} + C_{ff}\right)U^2 = \rho g\left(H_s + H_f\right)S$$

$$\tau_b = \rho C_f U^2 = \rho g H S$$

Skin friction + form drag



Skin friction only



The difference between the two characterizes form drag.

SKIN FRICTION AND FORM DRAG PREDICTORS

Skin friction

$$C_{fs} = \left[\frac{1}{\kappa}\ell n\left(11\frac{H_s}{k_s}\right)\right]^{-2} \quad \text{or} \quad C_{fs} = 0.0152\left(\frac{H_s}{k_s}\right)^{-1/3}$$

Form drag: empirical relation of Engelund and Hansen (1972)

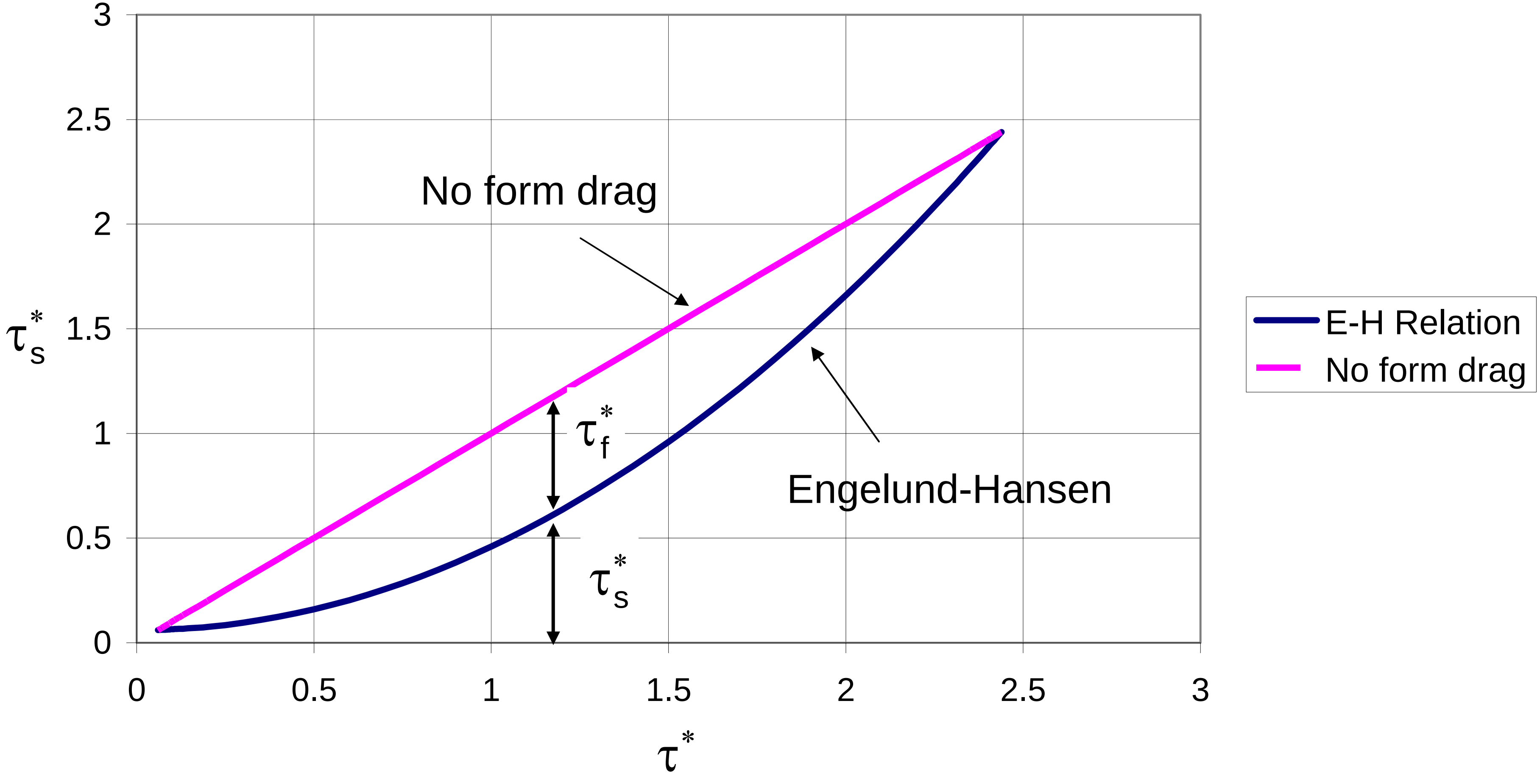
Total Shields number | *Shields number due to skin friction only*

$$\tau^* \equiv \frac{\tau_b}{\rho RgD} = \frac{u_*^2}{RgD} \qquad \tau_s^* \equiv \frac{\tau_{bs}}{\rho RgD} = \frac{u_{*s}^2}{RgD}$$

$$\tau_s^* = 0.06 + 0.4\left(\tau^*\right)^2$$

Note that bedforms are absent (skin friction only) when $\tau_s^* = \tau^*$;
bedforms are present when $\tau_s^* < \tau^*$;

Engelund-Hansen Bedform Resistance Predictor



FORM DRAG contd.

Engelund-Hansen tends to work well for sand-bed streams at laboratory and small field scale, but work poorly for large, low-slope sand-bed rivers.

Wright and Parker (2003) have modified it to accurately cover the entire range.

$$\tau_s^* = 0.05 + 0.7\left(\tau^* \mathbf{Fr}^{0.7}\right)^{0.8}$$

$$\mathbf{Fr} = \frac{U}{\sqrt{gH}} =$$ Froude number

In Wright-Parker applied to large sand-bed streams, dunes do not wash out at flood flows.

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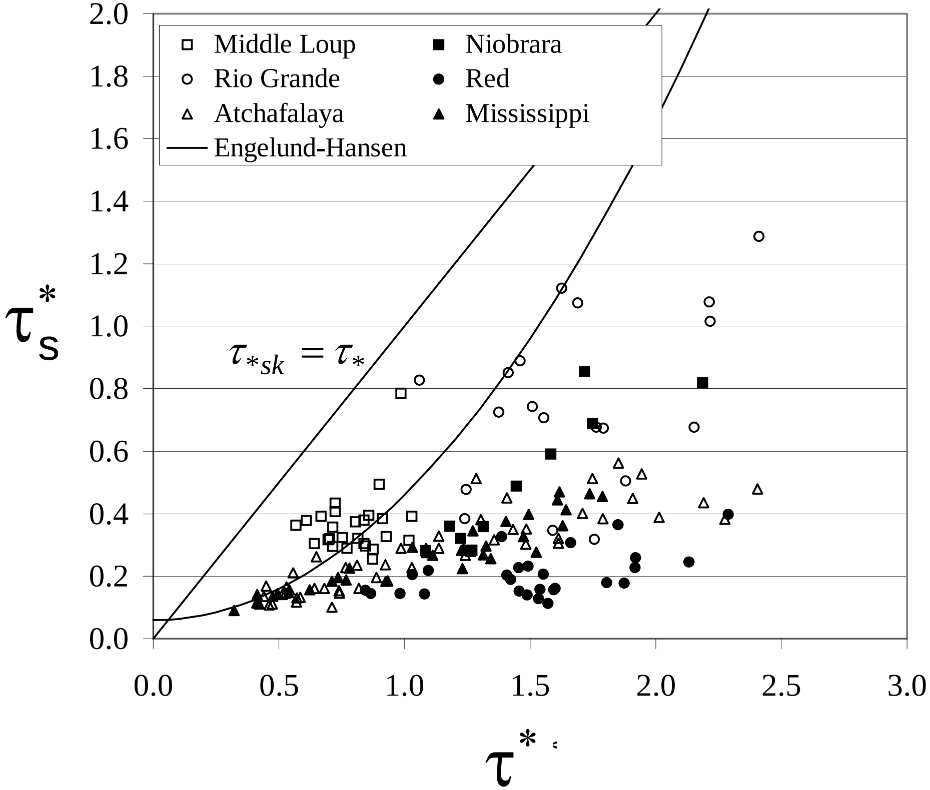
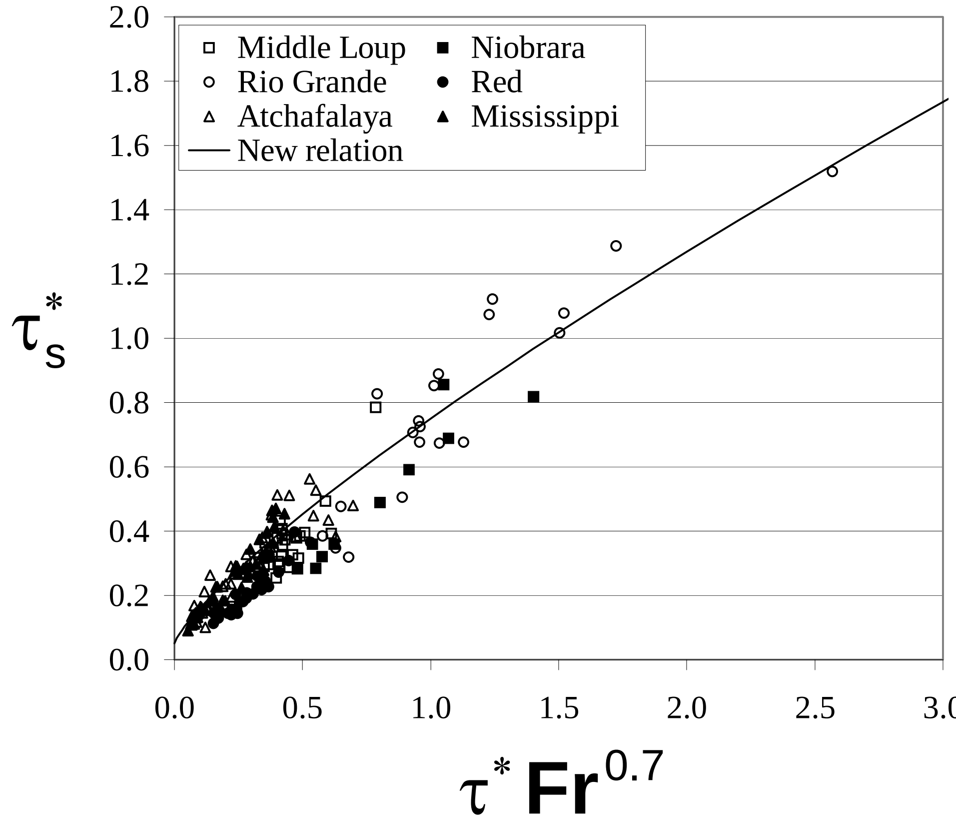




RELEVANCE OF DECOMPOSITION TO THE PREDICTION OF SEDIMENT TRANSPORT AND MORPHODYNAMICS

It all depends on what you want to do.

•In calculating bedload transport and entrainment into suspension over a **flat bed,** you can use the total boundary shear stress τ_b.

•In a stability analysis to explain the formation of **small scale bedforms (**see lectures of Blondeaux), the base state is a **flat bed**.

•If the bed is covered with dunes, the total boundary shear stress τ_b must be decomposed into skin friction τ_{bs} and form drag τ_{bf}, and **only the skin friction should be used** in computations of bedload transport and entrainment into suspension.

•In a stability analysis to explain the formation of **large scale bedforms** (e.g. bars on the continental shelf), the base state often includes small-scale bedforms, requiring a boundary shear stress decomposition.

RELATIONS FOR BEDLOAD TRANSPORT

THE SIMPLEST CASE IS 1D BEDLOAD TRANSPORT OVER A BED FOR WHICH BED SLOPE S IS NOT TOO LARGE (< 0.05?)

The formulations are long, so only essential results are quoted here.
$q_{bx} \rightarrow q_b$ = volume bedload transport per unit width [L^2T^{-1}].
$g_b = \rho_s q_b$ = mass bedload transport per unit width [L^2T^{-1}].

The archetypal form of bedload transport relations.

$$q_b^* = \frac{q_b}{\sqrt{RgD}\,D} =$$

Dimensionless Einstein bedload number

$$q_b^* = q_b^*(\tau_s^*)$$

Einstein bedload number is a monotonically increasing function of the Shields number due to skin friction.

THE MOTHER OF ALL MODERN BEDLOAD TRANSPORT RELATIONS: Meyer-Peter and Müller (1948)

Original (and very famous) form:

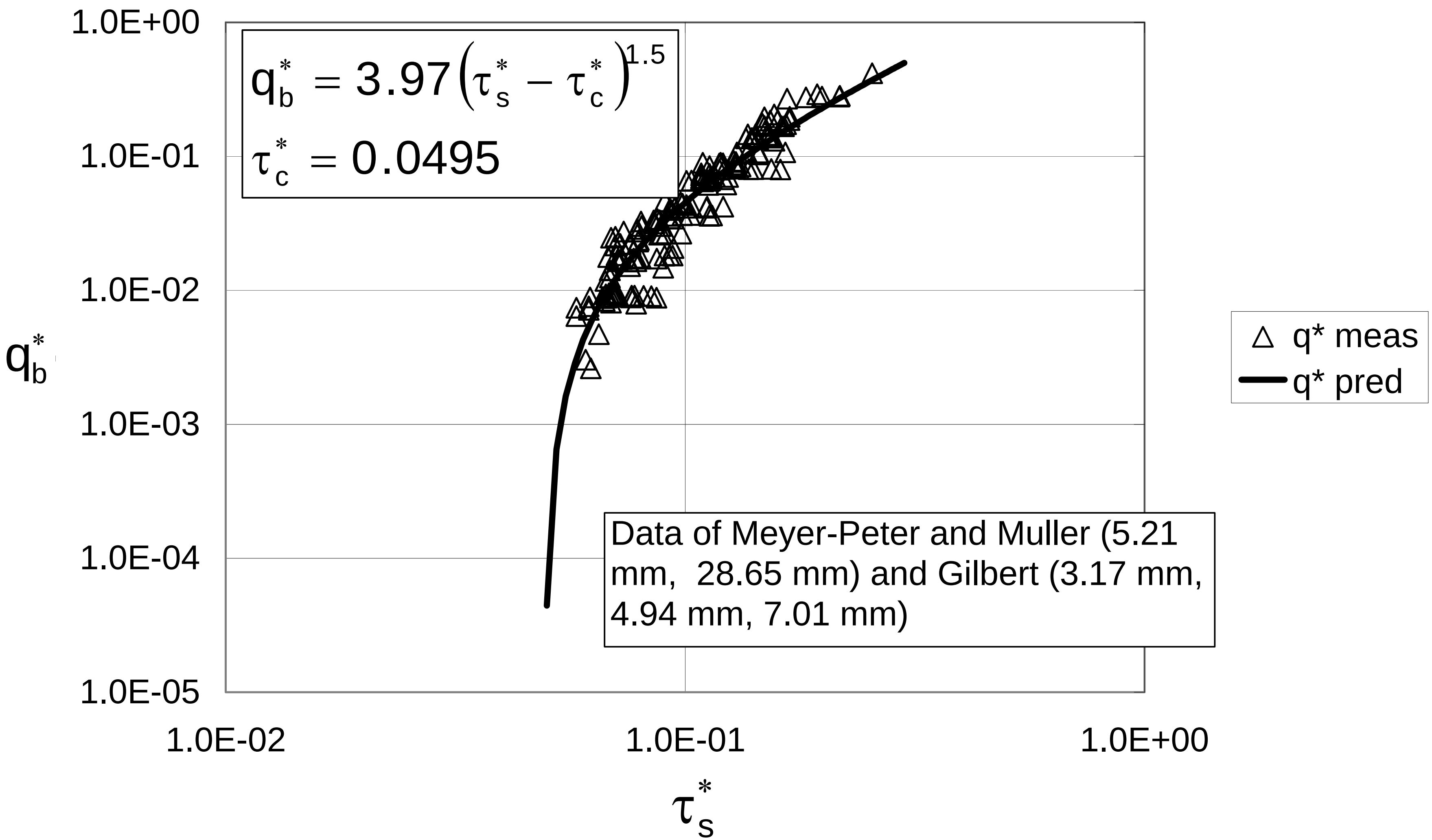
$$q_b^* = 8\left(\tau_s^* - \tau_c^*\right)^{1.5}, \quad \tau_c^* = 0.047$$

Corrected form due to Wong and Parker (2003)

$$q_b^* = 3.97\left(\tau_s^* - \tau_c^*\right)^{1.5}, \quad \tau_c^* = 0.0495$$

In both cases τ_c* is determined from data-fitting, and is much higher than an realistic threshold condition for motion.

Bedload Relation: Modified MPM



A SMORGASBORD OF BEDLOAD RELATIONS

$$q_b^* = 5.7\left(\tau_s^* - \tau_c^*\right)^{1.5}, \quad \tau_c^* = 0.03 \sim 0.042$$

Fernandez Luque & van Beek (1976)

$$q_b^* = 17\left(\tau_s^* - \tau_c^*\right)\left(\sqrt{\tau_s^*} - \sqrt{\tau_c^*}\right), \quad \tau_c^* = 0.05$$

Ashida & Michiue (1972)

$$q_b^* = 18.74\left(\tau_s^* - \tau_c^*\right)\left(\sqrt{\tau_s^*} - 0.7\sqrt{\tau_c^*}\right), \quad \tau_c^* = 0.05$$

Engelund & Fredsoe (1976)

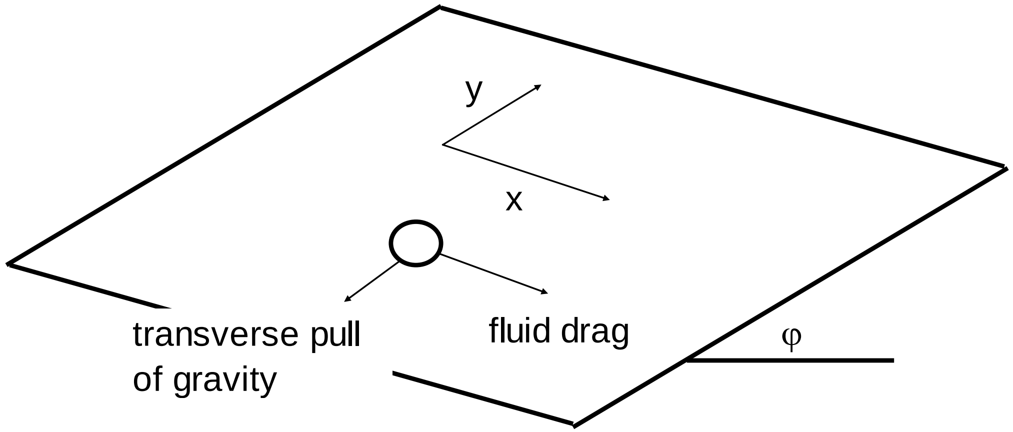
$$1 - \frac{1}{\sqrt{\pi}} \int_{-(0.143/\tau_s^*)-2}^{+(0.143/\tau_s^*)-2} e^{-t^2}\,dt = \frac{43.5 q_b^*}{1 + 43.5 q_b^*}$$

Einstein (1950)

$$q_b^* = 11.2\left(\tau_s^*\right)^{1.5}\left(1 - \frac{\tau_c^*}{\tau_s^*}\right)^{4.5}, \quad \tau_c^* = 0.03$$

Parker (1979) fit to Einstein (1950)

MILD TILT OF BED IN TRANSVERSE DIRECTION: BEDLOAD IS PULLED LATERALLY AS WELL AS DOWNSTREAM



Linearized formulation:

$$-\frac{\partial \eta}{\partial y} = \tan\varphi \cong \sin\varphi$$

In the absence of tilt the bedload transport vector is aligned with the vector of boundary skin friction.
In the presence of tilt the particles also are impelled by gravity down the slope

Archetypal form

$$\vec{q}_b = \left|\vec{q}_b\right| \left[\frac{\vec{\tau}_{bs}}{\left|\vec{\tau}_{bs}\right|} - \beta \left(\frac{\tau^*_{smag}}{\tau^*_c} \right)^{-n} \vec{\nabla}\eta \right]$$

$$\vec{\tau}_{bs} = (\tau_{bsx}, \tau_{bsy}) \, , \quad \tau^*_{smag} = \frac{\left|\vec{\tau}_{bs}\right|}{\rho RgD} \, , \quad \vec{\nabla}\eta = \left(\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y} \right)$$

Here β and n are constants

APPLICATION OF LINEARIZED FORMS

- Compute q_{bx} from an existing 1D model (but using critical Shields number that is modified for bed tilt.
- Compute q_{by} from the above value of q_{bx}, the vector of boundary skin friction and the transverse slope of the bed.

$$\left|\vec{q}_b\right| \cong q_{bx} \quad , \qquad \tau^*_{smag} \cong \tau^*_{sx}$$

$$q_{by} = q_{bx}\left[\frac{\tau_{bsy}}{\tau_{bsx}} - \beta\left(\frac{\tau^*_{sx}}{\tau^*_c}\right)^{-n}\frac{\partial\eta}{\partial y}\right]$$

$$\tau^*_{sx} = \frac{\tau_{bsx}}{\rho RgD}$$

SMORGASBORD OF RELATIONS FOR TRANSVERSE BEDLOAD TRANSPORT

μ_c = static Coulomb friction coefficient, $\mu_d < \mu_c$ = dynamic Coulomb friction coefficient

$$\beta = \frac{1}{\mu_d}, \quad n = 0$$

Engelund (1974)

$$\beta = \frac{1}{f_*}, \quad n = 1, \quad f_* = o(1) \text{ calib coeff}$$

Struiksma et al. (1985)

$$\beta = \frac{1}{\sqrt{\mu_c \mu_d}}, \quad n = \frac{1}{2}$$

Hasegawa (1984)

$$\beta = \frac{1 + r\mu_d}{\mu_d f_*}, \quad n = \frac{1}{2}, \quad r = 0.85, \quad f_* = 1.19$$

Johannesson & Parker (1989)

$$\beta = 0.75, \quad n = \frac{1}{4}$$

Sekine & Parker (1992)

FULLY NONLINEAR RELATIONS FOR THE TRANSPORT OF BEDLOAD ON ARBITRARILY TILTED BEDS

The formulation is too complicated to go into here.
But see:

Kovacs, A. and G. Parker, 1994, “A new vectorial bedload formulation and its application to the time evolution of straight river channels,” *J. Fluid Mech.*, 267, 153-183.

Parker, G., L. Solari, L. and G. Seminara, “Bedload at low Shields stress on arbitrarily sloping beds: alternative entrainment formulation,” *Water Resources Research*, in press (preprint available on request).

SHEET FLOW

•For values of $\tau_s^* <$ a threshold value τ_{sheet}^*, bedload is localized in terms of rolling, sliding and saltating grains that exchange only with the bed surface.

•When $\tau_s^* > \tau_{sheet}^*$ the bedload layer devolves into a sliding layer of grains that can be many grains thick. Sheet flows occur in unidirectional river flows as well as bidirectional flows in the surf zone.

•Values of τ_{sheet}^* have been variously estimated as 0.8 ~ 1.5. (Horikawa, 1988, Fredsoe and Diegaard, 1994, Dohmen-Jannsen, 1999)

SHEET FLOW

Experiments of Peng and Abrahams

video
clip