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NOTE ON THE ANALYSIS OF PLUNGING OF DENSITY FLOWS

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ABSTRACT

This note is devoted to the correction of an error in a calculation contained in a classical published paper on the plunging of river flows as they enter lakes or reservoirs. The correction of the error provides a felicitous result, according to which not only the ratio of underflow thickness just after plunging to depth just before plunging, but also the densimetric Froude numbers both just before and just after plunging are all specified as a function of a single parameter. This single parameter characterizes the tendency of plunging to entrain ambient water into the density underflow.

INTRODUCTION

When relatively denser river water meets relatively lighter ambient fluid as it flows into a lake or reservoir the dense river flow often plunges to form a bottom underflow. When the density difference is mediated by the presence of suspended mud in the water column of the river, the resulting underflow is termed a turbidity current. The phenomenon of plunging has been studied by a number of authors, including Singh and Shah (1971), Savage and Brimberg (1975), Jain (1981), Farrell and Stefan (1986) and Bournet et al. (1999).

Here attention is focused on the classical paper due to Akiyama and Stefan (1984), abbreviated to A&S in the text below. A&S offer a 1D formulation of plunging based on considerations of conservation of volume, mass and momentum. Momentum balance is imposed on two control volumes, referred to as Control Volume I and Control Volume II below.

The imposition of the conservation of volume, mass and momentum within Control Volume I results in a relation for the ratio H_d/H_p of the layer thickness of the underflow just after plunging H_d to the depth just before plunging H_p as a function of a) the densimetric Froude number just before plunging $\mathbf{Fr}_{dp}$ and b) a dimensionless parameter γ that describes the degree to which plunging entrains ambient water into the underflow.

The further imposition of momentum balance on Control Volume II adds a second relation, so that both $\mathbf{Fr}_{dp}$ and H_d/H_p, as well as the densimetric Froude number $\mathbf{Fr}_{dd}$ just downstream of plunging are all specified as functions of γ alone. A&S, however, missed this conclusion due to an error in their calculation. The error consists of the assumption of a critical value of $\mathbf{Fr}_{dd}$ (i.e. a value of unity when a relevant shape factor is also set equal to unity) in the case of plunging on

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a "steep" slope, and a normal value of $\mathbf{Fr}_{dd}$ in the case of plunging on a "mild" slope. These assumptions are not only erroneous but also unnecessary, for the elegant formulation of A&S already specifies $\mathbf{Fr}_{dd}$ independently.

In the present note the error is corrected, and a new set of relations for plunging is presented. The new relations are placed in both implicit analytical and explicit graphical form to allow for easy application.

INCOMPRESSIBILITY AND MASS BALANCE

The steady 1D plunging flow from a river into a body of water described in Figure 1 is considered. Dense river water flows from left to right into the ambient water of a lake, e.g. a reservoir. At point "pl" in the figure the river water plunges to form a dense underflow. The density of the river water, ρ_r is slightly greater than that of the ambient water in the lake, ρ_a. The depth and depth-averaged flow velocity in the river at section "p" just upstream of the plunge point are H_p and U_p, respectively. Just downstream of the plunge point at section "d" the layer thickness and layer-averaged velocity of the dense underflow are H_d and U_d, respectively. Plunging draws ambient water toward the plunge point at layer-averaged velocity U_a. The density of the underflow at section "d" is ρ_d, a value that is greater than ρ_a but less than ρ_r due to mixing of the ambient water and river water in the process of plunging. Points "p" and "d" are sufficiently close to each other to approximate the thickness of the ambient water at section "d" as equal to $H_p – H_d$.

Here plunging is treated as a shock in the same way as a hydraulic jump. The conditions of incompressibility (volume conservation) and mass conservation require the following respective conditions with respect to Control Volume I;

$$U_pH_p + U_a(H_p - H_d) - U_dH_d = 0 \tag{1}$$

$$\rho_rU_pH_p + \rho_aU_a(H_p - H_d) - \rho_dU_dH_d = 0 \tag{2}$$

The densities ρ_r and ρ_d are written as

$$\rho_r = \rho_a(1+\varepsilon_r) \quad , \quad \rho_d = \rho_a(1+\varepsilon_d) \tag{3a,b}$$

where $\varepsilon_r << 1$, $\varepsilon_d << 1$ and $\varepsilon_d < \varepsilon_r$. Defining a coefficient of mixing γ of ambient water into the underflow as

$$\gamma = \frac{U_a(H_p - H_d)}{U_pH_p} \tag{4}$$

it follows that Eq. (1) can be rewritten as

$$U_pH_p(1+\gamma) = U_dH_d \tag{5}$$

Eq. (2) reduces with Eqs. (3), (4) and (5) to the result

$$\varepsilon_d = \frac{\varepsilon_r}{1+\gamma} \tag{6}$$

The above formulation and results are identical to those appearing in A&S.

MOMENTUM BALANCE: CONTROL VOLUME I

Momentum balance over Control Volume I is now pursued. For simplicity the vertical profiles of density and velocity are assumed to be uniform in the river flow at section "p" just upstream of the plunge point, the dense underflow at section "d" just downstream of the plunge point, and the flow of ambient water at section "d" just downstream of the plunge point. These assumptions correspond to setting the shape factors S_1 and S_2 of A&S equal to unity, a condition that is easily relaxed if desired. In point of fact, Parker et al. (1987) found experimental values for S_1 and S_2 of 1.00 and 0.99, respectively, for the case of turbid underflows. The values, however, do not pertain to underflows immediately after plunging.

Momentum balance over Control Volume I requires that

$$\rho_a(1+\varepsilon_r)U_p^2H_p - \rho_a(1+\varepsilon_d)U_d^2H_d - \rho_a U_a^2(H_p - H_d) + F_{pp} - F_{pd} = 0 \qquad (7)$$

where F_{pp} and F_{pd} denote the pressure forces on the left- and right-hand sides of the control volume, respectively. These pressure forces are assumed to be hydrostatic, and are given by the relations

$$F_{pp} = \int_0^{H_p} p\big|_p dy \quad , F_{pd} = \int_0^{H_p} p\big|_d dy \qquad (8a,b)$$

where y denotes an upward vertical coordinate from the bed and p denotes pressure. The relations for pressure at sections "p" and "d" are, respectively

$$p\big|_p = \rho_a g(1+\varepsilon_r)(H_p - y)$$

$$p\big|_d = \begin{cases} \rho_a g(H_p - y) & , H_d < y < H_p \\ \rho_a g(H_p - H_d) + \rho_a(1+\varepsilon_d)(H_d - y) & , 0 < y < H_d \end{cases} \qquad (9a,b)$$

where g denotes the acceleration of gravity. Evaluating Eqs. (8a,b) with Eqs. 9(a,b), it is found that

$$F_{pp} = \frac{1}{2}\rho_a g(1+\varepsilon_r)H_p^2$$
$$F_{pd} = \frac{1}{2}\rho_a g\left(H_p^2 - H_d^2\right) + \frac{1}{2}\rho_a g(1+\varepsilon_d)H_d^2 \qquad (10a,b)$$

Substituting Eqs. (10a,b) into Eq. (7), and applying the standard Boussinesq approximation following from the assumptions $\varepsilon_r << 1$ and $\varepsilon_d << 1$, it is found that

$$U_p^2H_p - U_d^2H_d - U_a^2(H_p - H_d) + \frac{1}{2}\varepsilon_r g H_p^2 - \frac{1}{2}\varepsilon_d g H_d^2 = 0 \qquad (11a)$$

Reducing Eq. (11a) further with Eqs. (5) and (6), momentum balance in Control Volume I takes the form

$$U_p^2H_p - \frac{U_p^2H_p^2(1+\gamma)^2}{H_d} - \frac{U_p^2H_p^2\gamma^2}{H_p - H_d} + \frac{1}{2}\varepsilon_r g H_p^2 - \frac{1}{2}\frac{\varepsilon_r}{1+\gamma} g H_d^2 = 0 \qquad (11b)$$

The densimetric Froude number $\mathbf{Fr}_{dp}$ just upstream of the plunge point, densimetric Froude number $\mathbf{Fr}_{dd}$ just downstream of the plunge point and the ratio φ are defined as follows;

$$\mathbf{Fr}_{\mathbf{dp}}^{\mathbf{2}} = \frac{U_p^2}{\varepsilon_r g H_p} \quad , \quad \mathbf{Fr}_{\mathbf{dd}}^{\mathbf{2}} = \frac{U_d^2}{\varepsilon_d g H_d} \quad , \quad \varphi = \frac{H_d}{H_p} \qquad (12a,b,c)$$

Eq. (11) can be rewritten with Eqs. (12a) and (12c) in the form

$$\mathbf{Fr}_{dp}^2 - \mathbf{Fr}_{dp}^2 \frac{(1+\gamma)^2}{\varphi} - \mathbf{Fr}_{dp}^2 \frac{\gamma^2}{(1-\varphi)} + \frac{1}{2} - \frac{1}{2}\frac{\varphi^2}{(1+\gamma)} = 0 \qquad (13)$$

Between Eqs. (12a,b,c), (5) and (6) it is found that

$$\mathbf{Fr}_{dd}^2 = \mathbf{Fr}_{dp}^2 \frac{(1+\gamma)^3}{\varphi^3} \qquad (14)$$

Eq. (13) allows for the computation of the ratio φ as a function of the densimetric Froude number $\mathbf{Fr}_{dp}$ just upstream of plunging and the mixing ratio γ. That is, if ε_r, U_p, H_p and γ are specified, the values ε_d, U_d and H_d just downstream of plunging can be computed from Eq. (13) and Eqs. (5), (6), (12a) and (12c).

Eq. (13) specifies a fourth-order polynomial for φ. In the limiting case $\gamma \to 0$ it reduces to a third-order polynomial in φ, which is easily factored and solved to yield the result

$$\varphi = \frac{-1 + \sqrt{1 + 8\mathbf{Fr}_{dp}^2}}{2} \qquad (15)$$

i.e. a relation of precisely the same form as the one for conjugate depths of a hydraulic jump.

MOMENTUM BALANCE: CONTROL VOLUME II

As noted above, Eq. (13) allows the computation of the flow just downstream of the plunge point if $\mathbf{Fr}_{dp}$ and γ are specified. A&S go one step farther, however, to apply momentum balance to Control Volume II of Figure 1. Note that the left-hand side of Control Volume II is located at the point "pl," i.e. plunge point, rather than at "p" just upstream of it in Figure 1. As a result the flow velocity across the left-hand side of Control Volume II can be set equal to zero. In addition, A&S assume a hydrostatic pressure distribution on both sides of Control Volume II. This assumption is not likely to be completely accurate on the left-hand side, because the downwelling there creates a non-hydrostatic contribution to pressure. Assuming, however, that this non-hydrostatic component can be neglected, the following form for momentum balance is obtained;

$$-\rho_a U_a^2 (H_p - H_d) + \frac{1}{2}\rho_a (1+\varepsilon_r) g (H_p - H_d)^2 - \frac{1}{2}\rho_a g (H_p - H_d)^2 = 0 \qquad (16a)$$

or upon reduction

$$-U_a^2 (H_p - H_d) + \frac{1}{2}\varepsilon_r g (H_p - H_d)^2 = 0 \qquad (16b)$$

Further reducing Eq. (16b) with Eq. (4),

$$\frac{U_p^2 H_p^2 \gamma^2}{(H_p - H_d)} = \frac{1}{2} \varepsilon_r g (H_p - H_d)^2 \tag{17}$$

Between Eqs. (12a, 12c) and (17) the densimetric Froude number just upstream of the plunge point is evaluated as

$$\mathbf{Fr}_{dp}^2 = \frac{1}{2\gamma^2}(1-\varphi)^3 \tag{18}$$

Substituting Eq. (18) into Eq. (13), the following relation is found for the ratio φ;

$$\frac{1}{\gamma^2}(1-\varphi)^3 - \frac{1}{\gamma^2}\frac{(1-\varphi)^3}{\varphi}(1+\gamma)^2 - (1-\varphi)^2 + 1 - \frac{\varphi^2}{(1+\gamma)} = 0 \tag{19}$$

The addition of the momentum balance of Control Volume II is seen to change the character of the problem so as to provide extra information. In the case of Eq. (13), it was necessary to specify both $\mathbf{Fr}_{dp}$ and γ (or equivalently $\mathbf{Fr}_{dd}$ and γ) in order to compute φ. It is seen from Eqs. (18) and (19), however, that both $\mathbf{Fr}_{dp}$ and φ can be computed once γ is specified. That is, the addition of Eq. (17) allows for direct computation of the densimetric Froude number just upstream of plunging.

Rewriting Eq. (19) with the definition

$$\chi = 1 - \varphi \tag{20}$$

it is found that

$$\chi^3 - \frac{\chi^3}{1-\chi}(1+\gamma)^2 - \chi^2\gamma^2 + \gamma^2 - \frac{\gamma^2}{1+\gamma}(1-\chi)^2 = 0 \tag{21}$$

again yielding a fourth-order polynomial for χ. In the limit as $\gamma \to 1$ the only solution to Eq. (21) is

$$\chi = 0 \tag{22}$$

or thus

$$\varphi = 1 \quad , \quad H_d = H_p \tag{23a,b}$$

i.e. no plunging. That is, Eq. (21) implies that plunging is impossible in the absence of mixing. Substituting this result directly into Eq. (18) yields an indeterminate result for the densimetric Froude number $\mathbf{Fr}_{dp}$ just upstream of plunging.

The indeterminancy can be resolved by seeking an asymptotic expansion in small γ of the solution to Eq. (21) of the form

$$\chi = a\gamma^m + ... \tag{24}$$

Substituting Eq. (22) into Eq. (21), the appropriate form is found to be

$$\chi = 2^{1/3}\gamma^{2/3} + ... \tag{25}$$

which when substituted into Eq. (18) yields the result

$$\ell\text{im}_{\gamma\to 0}\mathbf{Fr}_{dp} = 1 \tag{26}$$

RELATIONS FOR PLUNGING

The authors were not able to factor, and thus solve Eq. (21) analytically for the case of finite γ. The equation is, however, readily solved numerically. Figure 2 shows the results of this solution, augmented with Eqs. (14) and (18), to yield predictions of φ, $\mathbf{Fr}_{dp}$ and $\mathbf{Fr}_{dd}$ as functions of the mixing coefficient γ.

The results appear eminently reasonable. For the case $\gamma = 0$ the expected result is obtained: $\mathbf{Fr}_{dp}$ and $\mathbf{Fr}_{dd}$ are both equal to unity. $\mathbf{Fr}_{dp}$ then decreases monotonically below unity as γ increases. Over the range $0 < \gamma < 0.14$ it is seen that $\mathbf{Fr}_{dd}$ is modestly less than unity. At a value of γ near 0.14 it again rises to unity. Over the range $\gamma > 0.14$ $\mathbf{Fr}_{dd}$ increases monotonically with increasing γ above unity. The analysis indicates that plunging to supercritical flow is not possible unless the mixing coefficient γ is sufficiently large.

DISCUSSION

It is of value to recount in precise detail how the above results differ from A&S. The present paper is completely equivalent to A&S in regard to volume conservation [present Eq. (1) $\leftrightarrow$ A&S Eq. (4)], mass conservation [present Eq. (2) $\leftrightarrow$ A&S Eq. (5)], momentum conservation on Control Volume I [present Eq. (11a) $\leftrightarrow$ A&S Eq. (16)] and momentum conservation on Control Volume II (present Eq. (16a) $\leftrightarrow$ A&S Eq. (17)].

In the analysis of A&S, the present Eq. (16b) [A&S Eq. (17)] is used to eliminate only the term containing the ambient flow velocity U_a in Eq. (11a) [A&S Eq. (16)] to obtain the result

$$U_p^2 H_p - U_d^2 H_d + \varepsilon_r g H_p H_d - \frac{1}{2} g \varepsilon_r \left(1 + \frac{1}{1+\gamma}\right) H_d^2 = 0 \tag{27}$$

[A&S Eq. (18)]. A&S then solve for U_p in terms of U_d using Eq. (5), and reduce using the definition of Eq. (12b) for the densimetric Froude number just downstream of plunging and Eq. (6) to obtain the polynomial

$$\left(\frac{1}{\varphi}\right)^2 - \left(\frac{2+\gamma}{2} + \mathbf{Fr}_{dd}^2\right)\frac{1}{1+\gamma}\left(\frac{1}{\varphi}\right) + \frac{\mathbf{Fr}_{dd}^2}{1+\gamma} = 0 \tag{28}$$

[A&S Eq. (20), in which they denote $1/\varphi$ as the parameter K].

The above equation is completely consistent with the present analysis. A&S then make one of two assumptions in Eq. (28): that the densimetric Froude number just downstream of plunging $\mathbf{Fr}_{dd}$ is equal to the critical value on “steep” slopes and the value $\mathbf{Fr}_{dn}$ associated with normal flow on “mild” slopes, i.e. the choices

$$\mathbf{Fr}_{dd} = 1 \tag{29a}$$

or

$$\mathbf{Fr}_{\mathbf{dd}} = \mathbf{Fr}_{dn} \tag{29b}$$

corresponding to the assumption of shape factors S_1 and S_2 in A&S equal to unity [A&S Eq. (42) or (40), respectively with $S_1 = 1$ and $S_2 = 1$]. In the case of Eq. (29a) this allows for the solution of Eq. (28) in closed form, yielding

$$\frac{1}{\varphi} = \frac{1}{2(1+\gamma)}\left[\left(\frac{2+\gamma}{2}+1\right)+\sqrt{\left(\frac{2+\gamma}{2}+1\right)^2-\frac{4}{1+\gamma}}\right] \tag{30}$$

[A&S Eq. (46)]. The corresponding solution in the case of Eq. (29b) is A&S Eq. (45).

The error in the analysis of A&S is the assumption of Eqs. (29a,b), according to which the densimetric Froude number $\mathbf{Fr}_{dd}$. just downstream of plunging must be equal to either the critical or normal value depending on the slope. In point of fact the analysis of A&S itself does not allow an independent specification of $\mathbf{Fr}_{dd}$. Rather, $\mathbf{Fr}_{dd}$ must be related to $\mathbf{Fr}_{dp}$ according to Eq. (14), which must in turn be related to φ and γ according to Eq. (18). Using Eqs. (14) and (18) in conjunction with Eq. (28) results in Eq. (19), which specifies φ as a function of γ alone. Once φ is computed from this relation, $\mathbf{Fr}_{dp}$ is computed from Eq. (18) and $\mathbf{Fr}_{dd}$ is computed from Eq. (14). It is seen from Figure 2 that the only values of γ resulting in a value of $\mathbf{Fr}_{dd}$ of unity are the two choices 0 and 0.14.

Correction of the error, however, allows a felicitous conclusion. The basic structure of the analysis of A&S (1984), in terms of the conservation of volume, mass and momentum in Control Volume I and the conservation of momentum in Control Volume II is in fact sufficient to specify not only the relation between underflow thickness H_d just after plunging and the flow depth H_p just before plunging, but also the densimetric Froude numbers $\mathbf{Fr}_{dp}$ and $\mathbf{Fr}_{dd}$ just before and just after plunging, all as functions of a single dimensionless parameter γ characterizing mixing at the plunge point.

CONCLUSION

The correction of an error in the computations of Akiyama and Stefan (1984) has a most felicitous result. The basic structure of their analysis proves sufficient to specify all relevant parameters concerning 1D plunging, including: a) the ratio of underflow thickness just downstream to depth just upstream; and b) the densimetric Froude numbers just upstream and downstream, as functions of a single dimensionless parameter γ. This parameter characterizes the entrainment of ambient fluid into the underflow at the point of plunging.

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APPENDIX I.--REFERENCES

Akiyama, J., and Stefan, H. (1984). “Plunging flow into a reservoir: Theory.” *Journal of Hydraulic Engineerin*g, ASCE, 110(4), 484-499.

Bournet, P., Dartus, D, Tassin, B., and Vincon-Leite B. (1999). “Numerical Investigation of plunging density current.” *Journal of Hydraulic Engineering,* ASCE, 125(6), 584-594.

Farrell, G. J., and Stefan, H. (1986). “Mathematical modeling of plunging reservoir flows.” *Journal of Hydraulic Researc*h, 26(5), 525-537.

Jain, S. C. (1981). “Plunging phemonena in reservoirs.” *Proceeding*s, Symposium on Surface Water Impoundments, ASCE, Minneapolis, MN. U.S.A., June 2 – 5, 1980.

Parker, G., Garcia, M., Fukushima, Y. and Yu, W. (1987). “Experiments on turbiditry currents over an erodible bed.” *Journal of Hydraulic Research*, 25(1), 123-147.

Savage, S. B. and Brimberg, J. (1975). “Analysis of plunging phenomena in water reservoirs.” *Journal of Hydraulic Research*, 13(2), 187-205.

Singh, B., and Shah, C.R. (1971). “Plunging phenomenon of density currents in reservoirs.” *La Houille Blanche*, 26(1), 59-64.

APPENDIX II.--NOTATION

The following symbols are used in this paper:

a	=	coefficient in Eq. 24;
F_{pd}	=	pressure force just downstream of plunging, defined by Eq. 10b;
F_{pp}	=	pressure force just upstream of plunging, defined by Eq. 10a;
$\mathbf{Fr}_{dd}$	=	densimetric Froude number just upstream of plunging, defined by Eq. 12a;
$\mathbf{Fr}_{dp}$	=	densimetric Froude number just downstream of plunging, defined by Eq. 12b;
g	=	gravitational acceleration;
H_d	=	thickness of density underflow just downstream of plunging;
H_p	=	river depth just upstream of plunging;
m	=	exponent in Eq. 24;
p	=	pressure;
U_d	=	layer-averaged velocity of density underflow just downstream of plunging;
U_p	=	layer-averaged velocity of density underflow just upstream of plunging;
y	=	coordinate defined upward normal from the bed;
χ	=	$1 - \varphi = 1 - (H_d/H_p)$;
ε_d	=	fractional density excess in the underflow just downstream of plunging;
ε_r	=	fractional density excess in the river water just upstream of plunging;
γ	=	dimensionless mixing parameter defined by Eq. 4;
φ	=	the ratio H_d/H_p;
ρ_a	=	density of the ambient water in the lake or reservoir;
ρ_d		density of the underflow just downstream of plunging;

ρ_r = density of the river water just upstream of plunging.

FIGURE CAPTIONS

Figure 1. Definition diagram for plunging showing Control Volumes I and II.
Figure 2. Predictions for φ, $\mathbf{Fr}_{dp}$ and $\mathbf{Fr}_{dd}$, as functions of γ.

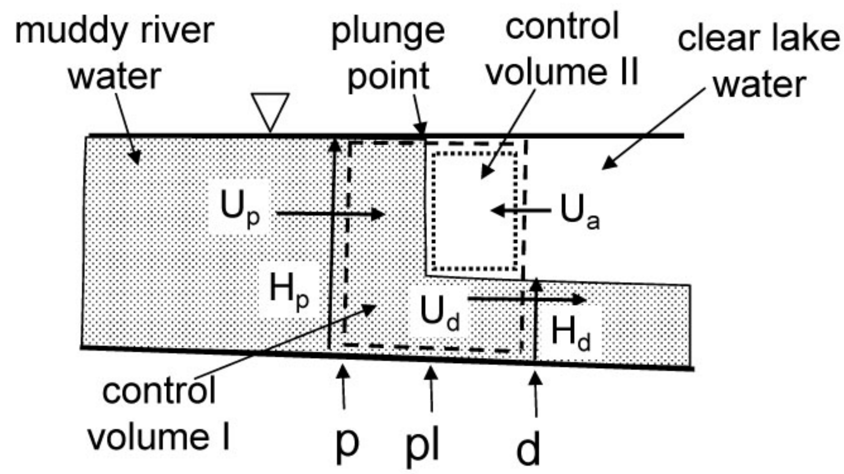

Figure 1

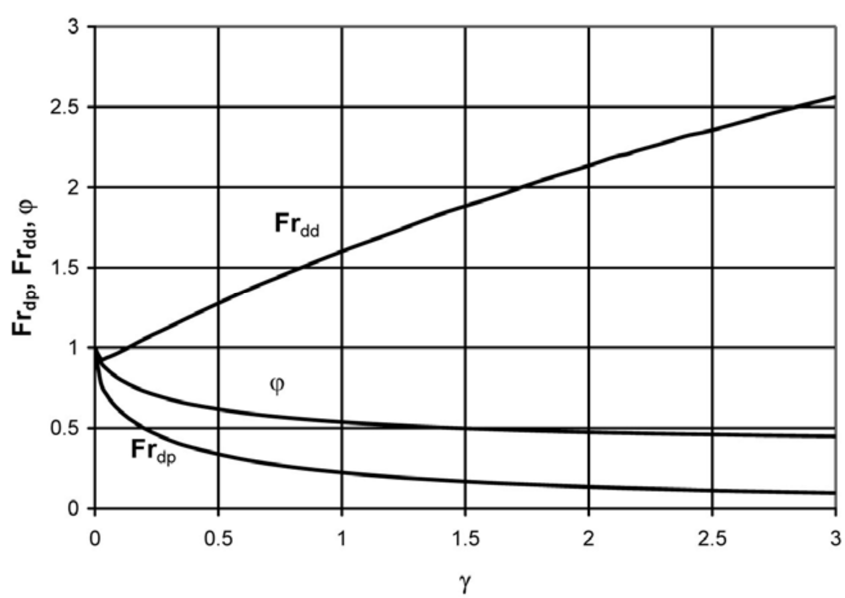

Figure 2