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PERSISTENCE OF SEDIMENT LUMPS IN APPROACH TO EQUILIBRIUM IN SEDIMENT-RECIRCULATING FLUMES

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ABSTRACT

Flumes for the study of mobile-boundary hydraulics are of two basic kinds: the sediment-recirculating and sediment-feed flumes. It is shown here that these two types approach mobile-bed equilibrium along very different paths. In the former case this path is characterized by recirculating lumps of sediment that gradually decay.

Keywords: sediment transport, flumes, equilibrium, morphodynamics.

INTRODUCTION

There are two types of sediment transport flumes, i.e. sediment-feed and sediment-recirculating. In the former water and sediment are introduced upstream and allowed to flow out freely downstream. In the latter water and sediment are recirculated from downstream to upstream with a pump. Both types are schematized in Fig. 1.

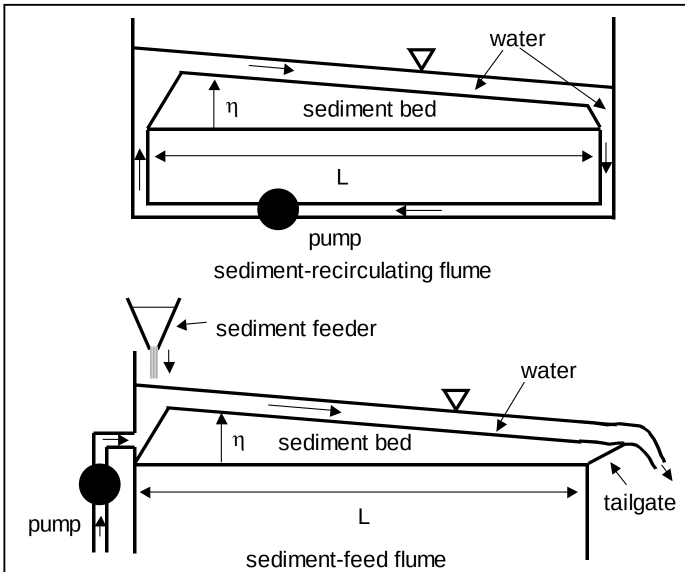

Figure 1. Sediment-recirculating and sediment-feed flumes

Flumes, as opposed to field rivers, eventually adjust to a mobile-bed equilibrium, at which flow and sediment transport become steady and uniform in the streamwise direction. The establishment of this equilibrium allows the evaluation of data for sediment transport relations.

Flow in a flume commences at conditions that differ from mobile-bed equilibrium, and then attains equilibrium over time. Although both recirculating and sediment-feed flumes can attain the same equilibrium, at least with uniform sediment, the approach is very different. Unpublished anecdotal reports suggest that in recirculating flumes migrating lumps, or waves of sediment often persist for long periods of time before equilibrium is achieved. Here the problem is pursued numerically with a generic model for both types of flumes.

GENERAL GOVERNING RELATIONS

The case considered is one for which not only the ultimate normal equilibrium, but also the approach to equilibrium, including the initial flow is everywhere Froude-subcritical. The flume has length L and constant width B. The streamwise coordinate x originates at the upstream end of the flume; t = time. Flow and sediment transport are approximated as 1D.

Water discharge per unit width = q_w and volume sediment transport rate per unit width = q. The flume contains uniform sediment of size D and density ρ_s.

The 1D equations of water mass and momentum balance are

$$\frac{\partial h}{\partial t}+\frac{\partial uh}{\partial x}=0 \quad , \quad \frac{\partial uh}{\partial t}+\frac{\partial u^2 h}{\partial x}=-\frac{1}{2}g\frac{\partial h^2}{\partial x}-gh\frac{\partial \eta}{\partial x}-\frac{\tau_b}{\rho} \qquad \text{(1a,b)}$$

where η = bed elevation, g = acceleration of gravity, u = depth-averaged flow velocity and τ_b = boundary shear stress at the bed, given by the relation

$$\tau_b = \rho C_f u^2 \qquad \text{(2)}$$

where ρ = water density and C_f = dimensionless bed resistance coefficient. The 1D Exner equation of sediment continuity takes the form

$$(1-\lambda_p)\frac{\partial \eta}{\partial t}=-\frac{\partial q}{\partial x} \qquad \text{(3)}$$

where λ_p denotes the porosity of the bed sediment.

In the case of subcritical flow for which the condition

$$q/q_w << 1 \qquad \text{(4)}$$

holds, and for which flow transients are not imposed externally, the above equations can be simplified with the quasi-steady assumption for morphodynamic evolution, according to which the time derivatives are dropped in (1a,b), yielding the forms

$$uh = q_w \quad , \quad \frac{\partial u^2 h}{\partial x}=-\frac{1}{2}g\frac{\partial h^2}{\partial x}-gh\frac{\partial \eta}{\partial x}-\frac{\tau_b}{\rho} \qquad \text{(5a,b)}$$

but retained in (3). Further reducing (5b) with the aid of (5a) and (2) results in

$$\frac{\partial h}{\partial x}=\left(-\frac{\partial \eta}{\partial x}-C_f\frac{q_w^2}{gh^3}\right)\Bigg/\left(1-\frac{q_w^2}{gh^3}\right) \qquad \text{(6)}$$

i.e. the standard 1D backwater equation. Here C_f is assumed to be a constant for simplicity.

Sediment transport is described in terms of a generic bed material load relation of the form

$$q^* = \begin{cases} 0 & if \quad \tau^* < \tau_c^* \\ \alpha_L(\tau^*-\tau_c^*)^{N_L} & if \quad \tau^* > \tau_c^* \end{cases} \qquad \text{(7)}$$

where α_L = dimensionless coefficient, N_L = dimensionless exponent and q^* = the Einstein number, defined as

$$q^* = \frac{q}{\sqrt{RgD}\,D} \quad , \quad R=\frac{\rho_s}{\rho}-1 \qquad \text{(8,9)}$$

In addition, τ^* denotes the Shields number, defined as

$$\tau^* = \frac{\tau_b}{\rho RgD} \qquad \text{(10)}$$

and τ_c^* denotes the critical Shields number for the onset of (significant) sediment transport. In the presence of significant bedform resistance the boundary shear stress τ_b must be replaced with only that due to skin friction τ_{bs}. This is not done here for simplicity.

FLUME CONSTRAINTS

Sediment-feed flume In a sediment-feed flume constant water discharge q_w is applied always and everywhere, and sediment feed rate q_f is prescribed at $x = 0$, yielding

$$q\big|_{x=0} = q_f \qquad \text{(11)}$$

The most common way to operate a sediment-feed flume is to adjust a tailgate so that the water surface elevation just upstream is maintained at a set value ξ_e, yielding;

$$\left(\eta + h\right)\Big|_{x=L} = \xi_e \tag{12}$$

Recirculating flume Here q_w is again a given constant everywhere and always. Assuming that a) end effects are negligible, b) the total amount of water stored in the recirculating pipe is constant and c) the total amount of sediment similarly stored is negligible, it follows that

$$\int_0^L h dx = C_1 \qquad \int_0^L \eta dx = C_2 \tag{13a,b}$$

where C_1 and C_2 = constants. These equations describe water and sediment conservation.

Integrating the Exner equation (3) over the length of the flume yields

$$(1-\lambda_p)\frac{d}{dt}\int_0^L \eta dx = q(0) - q(L) \tag{14}$$

Between (14) and (13b)

$$q(0) = q(L) \tag{15}$$

i.e. the boundary condition on sediment transport must be cyclic.

In fact the discharge constraint in a recirculating flume is not one of specified discharge, but rather one of specified pump head-discharge relation. An analysis of this case will be presented elsewhere; the results usually differ little from the case of specified discharge.

NORMAL OR EQUILIBRIUM STATE

At mobile-bed normal flow, water mass and momentum balance relations (5a,b) reduce to

$$q_w = h_n u_n \quad , \quad \tau_{bn} = \rho g h_n S_n \tag{16a,b}$$

where S = bed slope and the subscript "n" = normal flow. In addition, (2) and (7) apply to normal equilibrium as well as deviation from it. Reducing (16b) and (7) with (2) and (5a),

$$C_f q_w^2 = g h_n^3 S_n \quad , \quad q_n = \alpha_L \sqrt{RgD}\, D\left(\frac{h_n S_n}{RD} - \tau_c^*\right)^{N_L} \tag{17a,b}$$

Now if sediment size D and submerged specific gravity R are specified along with α_L, N_L and C_f, (17a,b) define two constraints on the four parameters q_w, q_n, h_n and S_n. That is, if any two of these parameters are specified then the other two can be computed.

In a sediment-feed flume the specified parameters are q_w and q_n, where

$$q_n = q_f \tag{18}$$

No matter what the initial conditions, the flow must eventually adjust to attain the values of h_n and S_n computed from the specified values q_w and q_n and (17a,b). In a recirculating system the specified parameters are q_w and h_n, where h_n is the spatially constant depth satisfying the integral constraint (13a). Again, no matter what the initial conditions, the flow must eventually adjust to attain the values of q_n and S_n computed from the specified values q_w and h_n and (17a,b).

Note that the bed slope at normal equilibrium S_n can be specified independently of the mean bed elevation $\overline{\eta}$ at that equilibrium, where

$$\overline{\eta} = \frac{1}{L}\int_0^L \eta dx \tag{19}$$

In a sediment-feed flume the constraint (12) of specified downstream water surface elevation determines the equilibrium value of $\overline{\eta}$. In a recirculating flume the corresponding constraint is integral sediment conservation, i.e. (13b).

NON-DIMENSIONALIZATION AND SOLUTION SCHEME

The following recipe is used to define hatted dimensionless parameters;

$$h = h_n \hat{h} \qquad q = q_n \hat{q} \qquad x = L\hat{x} \qquad t = (1-\lambda_p)\frac{S_n L^2}{q_n}\hat{t} \tag{20a,b,c,d}$$

Note that both $\hat{h}$ and $\hat{q}$ $\rightarrow 1$ as the flow converges to normal flow. Before making bed elevation dimensionless it is of value to decompose it into a flume-averaged component $\overline{\eta}$ defined by (19) and a deviation from this average η_d;

$$\eta = \overline{\eta} + \eta_d \tag{21}$$

Note that according to (19) and (21) the deviatoric bed elevation η_d must integrate to zero;

$$\int_0^L \eta_d \, dx = 0 \tag{22}$$

The parameters $\overline{\eta}$ and η_d are made dimensionless as follows;

$$\overline{\eta} = h_n \hat{\eta}_a(\hat{t}) \qquad \eta_d = S_n L \hat{\eta}_d(\hat{x},\hat{t}) \tag{23a,b}$$

in which case (22) takes the dimensionless form

$$\int_0^1 \hat{\eta}_d \, d\hat{x} = 0 \tag{23c}$$

Between (21) and (23) it is seen that

$$\eta = S_n L \hat{\eta} \quad , \quad \hat{\eta} = \hat{\eta}_d + \alpha_f \hat{\eta}_a \tag{24a,b}$$

where

$$\alpha_f = h_n / (S_n L) \tag{25}$$

denotes a dimensionless "flume number" that plays an important role in the morphodynamic evolution toward equilibrium in both the sediment-feed and recirculating cases.

With the normalizations (20) and (23) and the relation (17a), (6) takes the dimensionless form

$$\alpha_f \frac{\partial \hat{h}}{\partial \hat{x}} = \frac{-\frac{\partial \hat{\eta}_d}{\partial \hat{x}} - \hat{h}^{-3}}{1 - \boldsymbol{Fr}_n^2 \, \hat{h}^{-3}} \quad , \quad \boldsymbol{Fr}_n = \left(\frac{q_w^2}{g h_n^3}\right)^{1/2} \tag{26,27}$$

where $\boldsymbol{Fr}_n$ = Froude number at normal flow, here assumed to be less than unity.

A scaled bed slope $\hat{S}$ can be defined as

$$\hat{S} \equiv -\frac{d\hat{\eta}_d}{d\hat{x}} \tag{28a}$$

Between (20c), (21), (23b) and (28a) it can be shown that

$$\hat{S} = \frac{S}{S_n} \quad , \quad S = -\frac{\partial \eta}{\partial x} \tag{28b,c}$$

That is, the scaled slope $\hat{S}$ is equal to unity when normal conditions are achieved. Under the conditions of normal flow, (26) reduces with the aid of (28a) to

$$\hat{S}_n = 1 \tag{29}$$

Integrating (28a) with the aid of (29) and (23c), the deviatoric bed elevation profile at normal conditions $\hat{\eta}_{dn}$ is found to be

$$\hat{\eta}_{dn} = \frac{1}{2} - \hat{x} \tag{30}$$

The dimensionless form of the sediment transport relation (7) reduces with (2), (5c), (10) and (20) to

$$\hat{q} = \begin{cases} 0 \quad if \quad \hat{h}^{-2} < \left(\tau_r^*\right)^{-1} \\ \left[\dfrac{\hat{h}^{-2} - \left(\tau_r^*\right)^{-1}}{1 - \left(\tau_r^*\right)^{-1}}\right]^{N_L} \quad if \quad \hat{h}^{-2} > \left(\tau_r^*\right)^{-1} \end{cases} \tag{31}$$

where

$$\tau_r^* = \frac{\tau_n^*}{\tau_c^*} \quad , \quad \tau_n^* = \frac{\tau_{bn}}{\rho R g D} \tag{32a,b}$$

Note here that $\tau_n{}^*$ is the Shields number associated with normal equilibrium. According to (31) $\hat{q} \to 1$ as $\hat{h} \to 1$, i.e. as mobile-bed equilibrium is approached.

Decomposing the bed elevation according to (24), the dimensionless form of the Exner equation of sediment continuity is found to be

$$\alpha_f \frac{d\hat{\eta}_a}{d\hat{t}} = \hat{q}\big|_{\hat{x}=0} - \hat{q}\big|_{\hat{x}=1}$$
$$\frac{\partial \hat{\eta}_d}{\partial \hat{t}} = -\frac{\partial \hat{q}}{\partial \hat{x}} - \left(\hat{q}\big|_{\hat{x}=0} - \hat{q}\big|_{\hat{x}=1}\right) \tag{33a,b}$$

The first of the above pair of equations describes the evolution of mean bed elevation toward equilibrium, and the second describes the corresponding evolution of deviatoric bed elevation.

The part of the dimensionless formulation peculiar to a recirculating flume is considered below. The constraint (13a) and (15) become

$$\int_0^1 \hat{h} d\hat{x} = 1 \quad , \quad \hat{q}\big|_{\hat{x}=1} = \hat{q}\big|_{\hat{x}=0} \tag{34,35}$$

in which case (33a,b) reduce to

$$\frac{d\hat{\eta}_a}{d\hat{t}} = 0 \quad , \quad \frac{\partial \hat{\eta}_d}{\partial \hat{t}} = -\frac{\partial \hat{q}}{\partial \hat{x}} \tag{36a,b}$$

A convenient datum for elevation in the case of a recirculating flume is the initial mean bed elevation. Because this value never changes in time according to (36a), a solution for deviatoric bed elevation completely describes the evolution toward equilibrium.

The part of the dimensionless formulation peculiar to a sediment-feed flume is considered below. The constraint (11) combined with the condition that the normal sediment transport rate q_n must be equal to the feed rate q_f in a sediment feed flume leads to the condition

$$\hat{q}\big|_{\hat{x}=0} = 1 \tag{37}$$

Before making (12) dimensionless, one must define a convenient elevation datum in the sediment-feed case, here chosen to be the average bed elevation $\overline{\eta}_n$ at normal conditions. The downstream water surface elevation ξ_e relative to this datum is given as

$$\xi_e = -\frac{1}{2} S_n x + h_n \tag{38}$$

as shown in Fig. 2. Since the value of ξ_e of (38) must be maintained at all flows, and not just normal flows, the constraint (12) reduces with the aid of (20a), (23) and (38) to

$$\hat{h}\Big|_{\hat{x}=1} = 1 - \hat{\eta}_a - \frac{1}{\alpha_f}\left(\frac{1}{2} + \hat{\eta}_d\big|_{\hat{x}=1}\right) \tag{39}$$

Equations (33a,b) reduce to the following forms for the case of sediment feed;

$$\alpha_f \frac{d\hat{\eta}_a}{d\hat{t}} = 1 - \hat{q}\big|_{\hat{x}=1} \quad , \quad \frac{\partial \hat{\eta}_d}{\partial \hat{t}} = -\frac{\partial \hat{q}}{\partial \hat{x}} - \left(1 - \hat{q}\big|_{\hat{x}=1}\right) \tag{40a,b}$$

The above relations highlight an interesting difference in the way that recirculating and sediment-feed flumes approach equilibrium. Since the mean bed elevation of a recirculating flume is fixed, the evolution is entirely "rotational" and expressed solely in terms of $\hat{\eta}_d$ In the sediment feed case "rotation" is accompanied by vertical "translation" of the mean bed elevation $\hat{\eta}_a$ as the bed adjusts to match the imposed downstream water surface elevation.

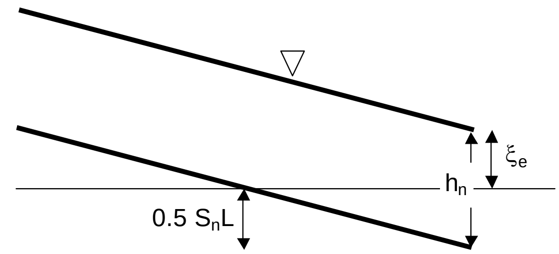

Figure 2. Definition diagram

The dimensionless parameters α_f, $\boldsymbol{Fr}_n$ and τ_r^* must be known in order to solve for the evolution toward equilibrium for either type of flume. Suppose that at time $\hat{t}$ the bed elevation profile is known in terms of $\hat{\eta}_a(\hat{t})$ and $\hat{\eta}_d(\hat{x},\hat{t})$. If the downstream value of depth $\hat{h}\big|_{\hat{x}=1}$ is also known, (26) can be integrated upstream for a solution for $\hat{h}$. Once $\hat{h}$ is known $\hat{q}$ can be obtained from (31), and the bed can be allowed to evolve according to (33).

In the case of a sediment-feed flume, the evaluation of $\hat{h}\big|_{\hat{x}=1}$ is simple and direct; it is specified by (39). The appropriate relations for sediment continuity for this case are (40). In a recirculating flume, however, the correct value of $\hat{h}\big|_{\hat{x}=1}$ is the one that yields a solution of (26) for $\hat{h}$ that in turn satisfies the integral constraint (34). This requires an iterative approach outlined below. The appropriate relation for sediment continuity is (36b).

SHOOTING TECHNIQUE FOR RECIRCULATING FLUMES

A shooting technique allows an iterative solution for $\hat{h}$ in the case of a recirculating flume. The following abbreviation is introduced;

$$\hat{h}_e = \hat{h}\Big|_{\hat{x}=1} \tag{41}$$

For any given bed profile and any prescribed value of $\hat{h}_e$, (26) can be solved to yield

$$\hat{h} = \hat{h}(\hat{x},\hat{h}_e) \tag{42}$$

Among these solutions, however, there is only one that satisfies (34). To find it iteratively from any first guess, it is useful to define the variational parameter H;

$$H = \partial\hat{h}/\partial\hat{h}_e \tag{43}$$

The equation for H is found by taking the derivative of both sides of (26) in $\hat{h}_e$; the associated b.c. is found by taking the derivative of (41) in $\hat{h}_e$, so that the problem becomes

$$\alpha_f \frac{\partial \hat{h}}{\partial \hat{x}} = \frac{-\dfrac{\partial \hat{\eta}_d}{\partial \hat{x}} - \hat{h}^{-3}}{1 - \boldsymbol{Fr}_n^2 \hat{h}^{-3}} \qquad \hat{h}(1) = \hat{h}_e$$

$$\alpha_f \frac{\partial H}{\partial \hat{x}} = \frac{3\hat{h}^{-4}}{1 - \boldsymbol{Fr}_n^2 \hat{h}^{-3}} [1 + \boldsymbol{Fr}_n^2 \frac{\dfrac{\partial \hat{\eta}_d}{\partial \hat{x}} + \hat{h}^{-3}}{1 - \boldsymbol{Fr}_n^2 \hat{h}^{-3}}] H \qquad H(1) = 1 \tag{44a,b,c,d}$$

The condition (34) becomes

$$\varphi(\hat{h}_e) = \int_0^1 \hat{h}(\hat{x}, \hat{h}_e) d\hat{x} - 1 = 0 \tag{45}$$

or invoking a Newton-Raphson scheme,

$$\hat{h}_e^{new} = \hat{h}_e - \frac{\varphi(\hat{h}_e)}{\varphi'(\hat{h}_e)} \qquad \varphi'(\hat{h}_e) = \frac{\partial}{\partial \hat{h}_e} \int_0^1 \hat{h}(\hat{x}, \hat{h}_e) d\hat{x} = \int_0^1 H d\hat{x} \tag{46a,b}$$

The above problem can be solved iteratively for $\hat{h}_e$, and thus $\hat{h}$.

INITIAL CONDITION

The initial condition is specified in terms of the initial bed profile;

$$\eta\big|_{t=0} = \eta_I(x) \tag{47a}$$

or in dimensionless terms

$$\hat{\eta}_a\big|_{\hat{t}} = \hat{\eta}_{aI} \quad , \quad \hat{\eta}_d\big|_{\hat{t}=0} = \hat{\eta}_{dI}(\hat{x}) \tag{48a,b}$$

A simple and useful example of an initial condition is a bed with some initial slope S_I that is constant in space. Scaled initial slope $\hat{S}_I$ is then given according to (28b) as

$$\hat{S}_I = S_I / S_n \tag{49}$$

It further follows from (23c) and (28) that

$$\hat{\eta}_{dI} = \hat{S}_I (0.5 - \hat{x}) \tag{50}$$

Thus if $\hat{S}_I = 0.5$ the initial bed slope is spatially constant and = half the normal bed slope.

In a recirculating flume with the elevation datum equal to the mean initial bed elevation, $\hat{\eta}_{aI}$ = $\hat{\eta}_a$ = 0. In a sediment-feed flume, the value of $\hat{\eta}_{aI}$ must be specified as well. Recall that the datum for the sediment-feed case has been chosen to be equal to the mean bed elevation at normal equilibrium, implying that $\hat{\eta}_{an}$ vanishes. Even if $\hat{\eta}_{aI}$ is also chosen to be vanishing, however, when $\hat{S}_I$ is not equal to $\hat{S}_n$ the parameter $\hat{\eta}_a$ may first deviate from zero before again approaching zero as equilibrium is approached

NUMERICAL IMPLEMENTATION

Space limits preclude the details of numerical implementation. The solution domain $0 \le \hat{x} \le 1$ is divided into $N - 1$ intervals bounded by N nodes, so that $\Delta\hat{x} = 1/(N-1)$. In addition, a ghost node is placed a distance $\Delta\hat{x}$ upstream of the origin. The ghost node is used only for implementing the Exner equation(s) of sediment continuity. Equation (26) for a sediment feed flume, or (44a,c) for a recirculating flume are solved by stepping upstream from $\hat{x} = 1$ using a predictor-corrector scheme. In the case of a recirculating flume the solution to (44a,c) is implemented iteratively. Once $\hat{h}$ is known at all nodes $i = 1$ to N $\hat{q}$ can be computed at these nodes from (31). A pure upwinding scheme, according to which

$$-\frac{\partial \hat{q}}{\partial \hat{x}} \cong \frac{\hat{q}_{i-1} - \hat{q}_i}{\Delta \hat{x}} \qquad (52)$$

is used for spatial derivatives; a central difference scheme yielded spurious fluctuations. The program implementing the solution is written in Visual Basic for Applications.

RESULTS

Space limitations preclude a thorough search of parameter space, so two characteristic examples are given here, one each for a sediment-recirculating flume (Fig. 3) and a sediment-feed flume (Fig. 4). The following values were used in the computations; $\boldsymbol{Fr}_n = 0.5$, $\tau_r^* = 3$, $\alpha_f = 10$, $\hat{S}_I = 0.5$, $N_L = 1.5$, $N = 50$, $\Delta\hat{t} = 0.0025$, $\hat{\eta}_{aI} = 0$ (sediment-feed case only). Let $\hat{S}_{top}$ denote the normalized bed slope based on the two nodes closest to the upstream end of the flume and $\hat{S}_{bot}$ denote the corresponding value based on the two nodes closest to the downstream end. The time development of $(\hat{S}_{bot}, \hat{S}_{top})$ is plotted in the figures. Both calculations begin at $(\hat{S}_{bot}, \hat{S}_{top})$ = (*0.5, 0.5*) (initial bed slope = half of equilibrium slope) and end at $(\hat{S}_{bot}, \hat{S}_{top})$ = (*1.0, 1.0*) (equilibrium). As time progresses, it is seen in Fig. 3 that the phase diagram spirals toward equilibrium, indicating the presence of recirculating sediment lumps that are damped in time. The dimensionless time required to reach an appropriate measure of equilibrium (a slope at every adjacent node pair differing by less than one percent from the equilibrium value) in the case of Fig. 3 is $\hat{t} = 9.95$. No such spiraling is seen in Fig. 4, where the time to reach the same measure of equilibrium is much shorter, i.e. $\hat{t} = 3.90$.

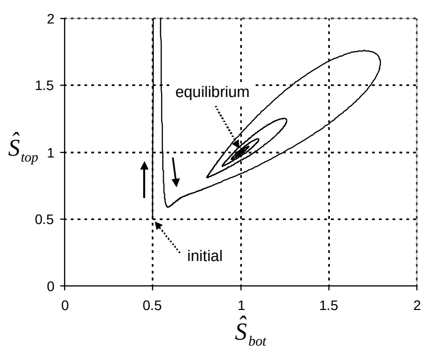

Figure 3. Approach to equilibrium; sediment-recirculating case

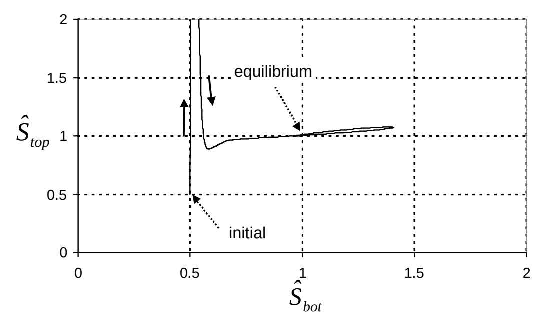

Figure 4. Approach to equilibrium; sediment-feed case

CONCLUSION

The results presented here show that sediment-feed flumes approach equilibrium along a markedly different path than sediment-recirculating flumes. The latter case is characterized by recirculating lumps of sediment that eventually decay. The cause of the lumps relates to the nature of recirculation, wherein the sediment supplied to the upstream end of the flume has nothing to do with the hydraulic conditions there, instead being determined by those at a distant point downstream. This disjoint is expressed in terms of recirculating lumps of sediment that decay slowly in time.

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