Transverse slope of bed and turbid-clear water interface of channelized turbidity currents flowing around bends

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ABSTRACT

Turbidity currents are sediment-laden bottom flows in lakes and the ocean that derive their momentum from the force of gravity acting on the sediment held in suspension. These currents are capable of creating highly meandering channels extending over hundreds or thousands of kilometers. The planforms of these channels show a remarkable resemblance to those of meandering rivers. In order to understand the flow of turbidity currents in sinuous channels a simple analysis of steady, streamwise-uniform flow around a bend of low, constant curvature is considered. The analysis is applied to both a river and a channelized turbidity current. In both cases the bed of the channel is sand, and both sand and mud are carried in suspension. The sand requires turbulence in order to be held in suspension, but the mud is sufficiently fine to allow it to be carried as wash load. The flow is assumed to be Froude-subcritical, and in the case of a turbidity current a relatively sharp interface between turbid water and clear water above is assumed. The analysis focuses on the processes that maintain a) the transverse water surface or interfacial slope associated with superelevation around bends and b) the transverse bed slope created by scour on the outside and fill on the inside of bends. In the case of rivers, the processes maintaining these two transverse slopes can be decoupled, allowing for the use of standard formulations. In a turbidity current, however, the processes may be strongly coupled, especially when the concentration of sand in suspension decreases strongly in the vertical.

1 INTRODUCTION

The most common river planform morphology is that of the meandering channel. A particularly striking image of a meandering river is that of Figure 1a. Here such meanders are referred to as “subaerial,” in that they are created by the interaction water flow and sediment transport in rivers under air (i.e. the atmosphere).


Figure 1a. Meandering river in Siberia. Photo courtesy A. Zaitsev.

The present paper is, however, concerned with another type of meandering channel, one with a striking resemblance to river meanders but in a radically different environment. The channels in question can be found at the bottom of the ocean and some lakes. They are formed by the action of turbidity currents, i.e. sediment-laden bottom currents that derive their driving force from the excess weight due to the presence of sediment. For this reason these channels can be said to display "subaqueous" meandering, in that the flows that create them consist of turbid water immersed in clear water. When these channels are found in the ocean, they may be said to describe "submarine" meandering. An image of submarine meandering that is every bit as striking as subaerial meandering is shown in Figure 1b. Here the term "submarine meandering" is used so as to include deep-water meandering channels in lakes as well.

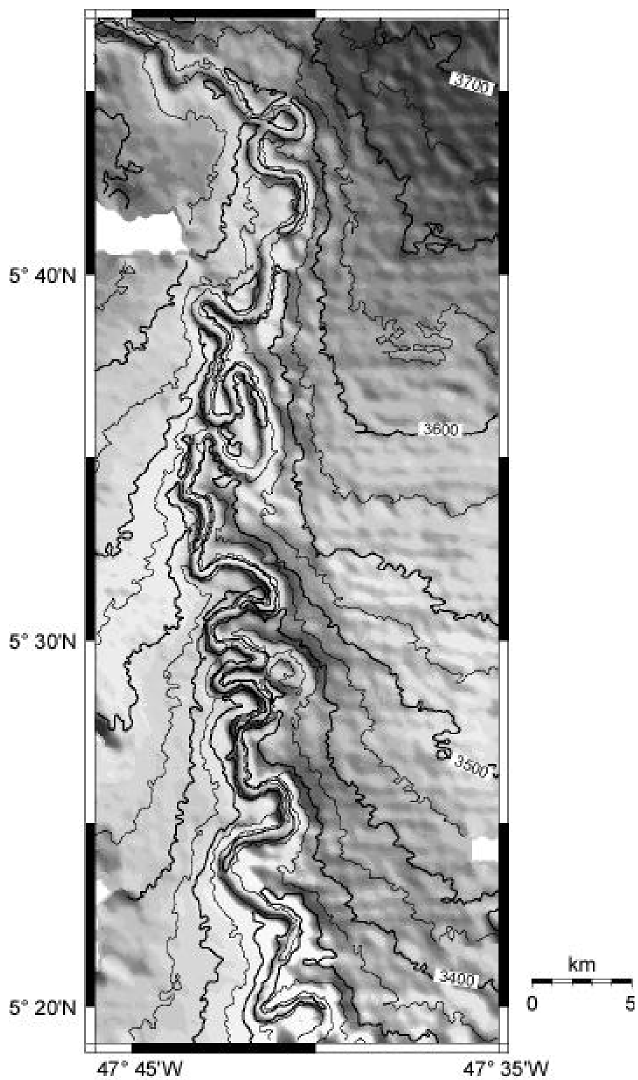

Figure 1b. Meandering channel on the Amazon submarine fan. From Pirmez (1994)

Submarine meandering channels possess both remarkable similarities to and notable differences from subaerial meandering channels. Here the analysis is focused on a relatively simple configuration: steady flow around a bend of constant curvature that is also uniform in the streamwise direction. Results are presented in parallel for both rivers and channelized turbidity currents.

The channel configuration to be studied is shown schematically in Figure 2, in which s denotes the streamwise direction, n denotes the transverse direction (directed such that n is positive on the outside of the bend) and z denotes a coordinate upward normal (nearly vertical) from the bed. The bend has constant centerline radius of curvature r_o and channel half-width b.

In the case of rivers it is well known that the flow superelevates around bends. That is, the centrifugal force generated by the bend is balanced by an air-water interface, or water surface that is high on the outside and low on the inside. The magnitude of the resulting transverse water surface slope is denoted on Figure 2 as S_{tt}. In addition, the secondary flow associated with a bend causes bed scour on the outside of the bend and deposition on the inside. The magnitude of this transverse bed slope is denoted on Figure 2 as S_{tb}.

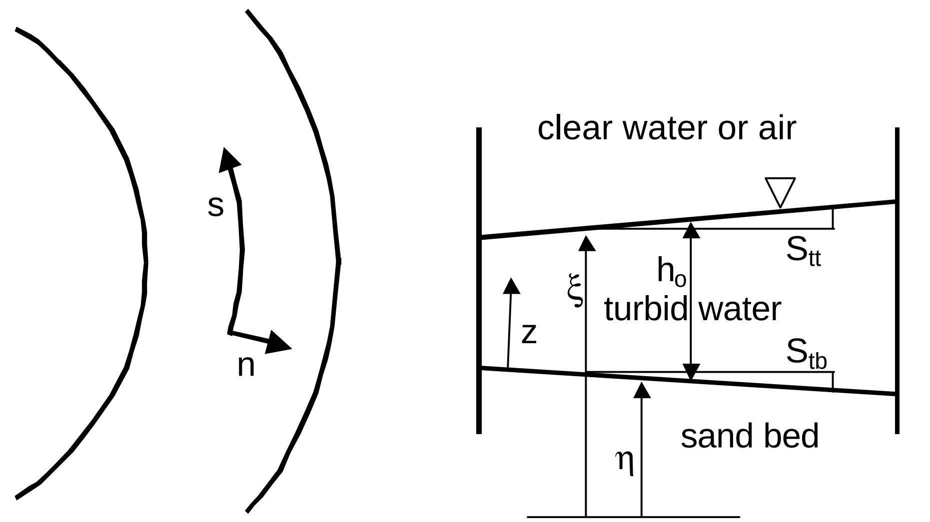

Figure 2. Planform (left) and cross-section (right) definition diagrams.

Standard relations exist in the literature for the prediction of S_{tt} and S_{tb}. They can be summarized in the forms

$$S_{tt} = \alpha \frac{h}{r_o} \mathbf{Fr}^2 \qquad \mathbf{Fr} = \frac{\bar{u}}{\sqrt{gh}}$$
$$S_{tb} = \frac{h}{r_o} A \tag{1a,b,c}$$

where h denotes flow depth, $\bar{u}$ denotes depth-averaged streamwise (primary) flow velocity, g denotes the acceleration of gravity, **Fr** denotes the Froude number, α is a parameter very near unity and A is an order-one scour parameter that is a function of the strength of the primary flow.

There is every reason to believe that similar processes operate in the case of channelized turbidity currents as they traverse bends. An example is given in Figure 3, which shows an acoustic image of a cross-section of a submarine channel with a turbidity current flowing in it. The image is from Hay (1987); the meandering channel was formed in engineering time by a turbidity current generated by the disposal of tailings from a mine. In the case of a turbidity current it is the interface between the turbid water below and the clear water above that is superelevated around bends. The expected patterns of both superelevation and bed scour are evident in Figure 3.

In the absence of better guidance, researchers on submarine meandering have tended to apply (1a-c) directly to channelized turbidity currents, with the one proviso that the Froude number is replaced with the densimetric Froude number (Komar, 1977; Pirmez, 1994; Imran et al., 1999). It is demonstrated here that such an extension is not generally possible. It is shown that in the case of rivers, the processes that maintain the transverse water surface slope are decoupled at linear level from those that maintain the transverse bed slope. In the case of turbidity currents, however, the processes that maintain the transverse interfacial slope are strongly coupled to those that maintain the transverse bed slope.

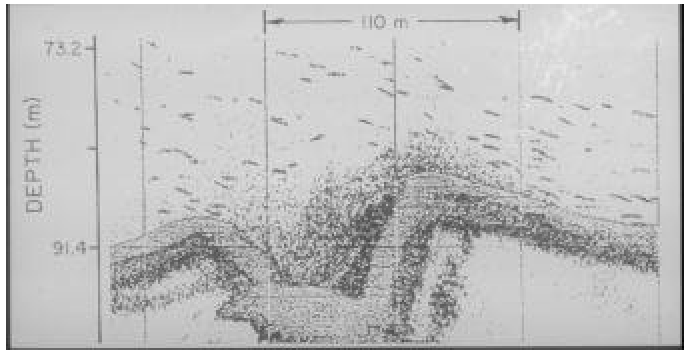

Figure 3. Acoustic image of the cross-section of a submarine channel near the apex of a bend showing superelevation of a turbidity current The outside of the bend is to the right (Hay, 1987).

The analysis presented here is not a complete solution, in that kinematic eddy viscosity is not dynamically tied to other parameters by means of a turbulence closure theory. This has the advantage of rendering the results independent of any particular closure formulation. It has the disadvantage, however, of yielding general rather than specific results. A sample implementation in the case of rivers is given using the algebraic closure of Engelund (1974). An appropriate closure for the case of channelized turbidity currents may be the formulation of Mellor and Yamada (1974).

Here the river or turbidity current is assumed to be flowing over a bed composed entirely of a uniform sand of size D_s and fall velocity v_s. This sand may be moved as bedload and suspended load. In addition, the flow carries mud in suspension as wash load, which neither deposits on nor is retained in the bed. The volume concentration of suspended sand and mud is assumed to be small everywhere.

In both cases the flows of interest are Froude-subcritical, in the standard sense for a river and in the densimetric sense for a turbidity current. This assumption is particularly important for the case of turbidity currents. The meandering channels created by turbidity currents on the bottom of the ocean often extend for hundreds or thousands of kilometers. This indicates that the turbidity current is able to follow the channel of its creation over long distances (Parker et al., 2000). Froude-supercritical turbidity currents should entrain ambient water from above, thus thickening in the downstream direction. It can be expected that the turbidity current would lose track of the channel once it becomes many times thicker than the channel depth. It follows that turbidity currents that create long meandering channels should likely be Froude-subcritical, and indeed sufficiently subcritical so that water entrainment at the interface can be neglected.

Here it is assumed that the turbidity current maintains a relatively sharp interface with the clear water above, as shown in Figure 4. Such sharp interfaces have been demonstrated experimentally for subcritical turbidity currents by Garcia and Parker (1989). The presence of mud with a negligible fall velocity can help facilitate such a sharp interface.

As a result of the above assumptions, it is assumed that the mud is distributed nearly uniformly in the vertical in both the river and the turbidity current, whereas the sand is sufficiently heavy so as to yield a concentration profile that decreases monotonically upward.

2. SETUP

The case under consideration is that of a curved channel with constant half-width b and radius of curvature in the streamwise direction, as illustrated in Figure 2. The bottom of the channel has constant transverse slope S_{tb}. The sidewalls of the channel are taken to be vertical and infinitely high. Channel curvature is assumed to be small compared to half-width; that is,

$$\frac{b}{r_o} << 1 \tag{2a}$$

The above assumption allows a treatment that is linearized for small curvature. The basis for the analysis presented here is the linear formulation of Parker and Johannesson (1989) for rivers.

In both the subaerial and submarine cases the flow is approximated to be steady and uniform in the streamwise and normal directions. Let h_o denote the centerline depth of flow in the channel in the subaerial case, and the centerline thickness of flow from bed to interface in the submarine case. It is further assumed that the flow is slender in the sense that

$$\frac{h_o}{b} << 1 \tag{2b}$$

The above approximation allows the neglect of sidewall effects everywhere in the flow except in the immediate vicinity of the sidewalls themselves.

As flow traverses the bend, centrifugal effects drive a superelevation of flow on the outside of the bend. Here this superelevation is simply described in terms of a top transverse slope S_{tt}.

The suspension is taken to be dilute. Let ρ denote the density of clear water, ρ_s denote the material density of sediment (sand or mud) and c_s and c_m denote the volume concentration of sand and mud, respectively, averaged over turbulence. It is thus assumed that

$$c_s + c_m << 1 \tag{3}$$

Let *R* denote the submerged specific gravity of the sediment, here defined as

$$R = \frac{\rho_s}{\rho} - 1 \tag{4}$$

For natural sediment *R* is typically close to 1.65. In a river the downslope driving force per unit weight of flow mixture is given as

$$gS_s[1 + R(c_s + c_m)] \cong gS_s \tag{5}$$

where g denotes the acceleration of gravity and S_s denotes the downstream bed slope. The approximation is valid for a dilute suspension. The turbidity current, however, flows only because of the presence of suspended sediment, which renders it heavier than the surrounding clear ambient water. As a result the downslope driving force per unit weight is given by the term

$$RgS_s(c_s + c_m) \tag{6}$$

It is seen via a comparison of (5) and (6) that for the same bed slope and the same dilute concentrations of sand and mud the driving force of a turbidity current is vastly reduced compared to that of a river.

3. PRIMARY FLOW

Here the limiting case as $r_o \to \infty$, i.e. flow in a straight channel is considered. In both the subaerial and subaqueous cases the flow velocity averaged over turbulence is denoted as u(z). The vertical mixing of both streamwise momentum and suspended sediment is represented in terms of a kinematic eddy viscosity $\varepsilon(z)$. Here $\varepsilon(z)$ is taken to be an arbitrary function, which nevertheless must take substantially different forms in the subaerial and submarine cases. The specification of a form for ε requires a specific turbulence closure theory. The assumption that the eddy diffusivity of momentum and suspended sediment should be identical corresponds to a Schmidt number of unity.

3.1 River

The governing equation for streamwise, or primary momentum balance in a straight open channel is

$$0 = gS_s + \frac{d}{dz}\varepsilon(z)\frac{du_o}{dz} \tag{7a}$$

where the subscript "o" denotes the primary flow in a straight channel. Appropriate boundary conditions for any turbulence closure model can be expressed in terms of a slip velocity u_s specified at the bed (which is in turn a function of other parameters such as the boundary shear stress τ_{sbo}) and vanishing shear stress at the water surface;

$$u_o(0) = u_s \qquad \varepsilon\frac{du_o}{dz}\bigg|_{z=h_o} = 0 \tag{7b,c}$$

The boundary shear stress τ_{sbo} and shear velocity u_{*o} are given by the relations

$$\tau_{sbo} = \rho\varepsilon\frac{du_o}{dz}\bigg|_{z=0} \qquad u_{*o} = \sqrt{\frac{\tau_{sbo}}{\rho}} \tag{8a,b}$$

The conservation equation for sand can be expressed as

$$\frac{d}{dz}\left(v_s c_{so} + \varepsilon\frac{dc_{so}}{dz}\right) = 0 \tag{9a}$$

The corresponding boundary conditions are

$$-\varepsilon\frac{dc_{so}}{dz}\bigg|_{z=0} = v_s E \qquad \left(v_s c_{so} + \varepsilon\frac{dc_{so}}{dz}\right)\bigg|_{z=h_o} = 0 \tag{9b,c}$$

where E is a dimensionless entrainment rate of sand into suspension, typically expressed as a function of boundary shear stress τ_{sbo} and other parameters.

The corresponding formulation for conservation of suspended mud is obtained from the above equations by the transformation $c_{so} \to c_{mo}$ and the conditions $E = 0$, $v_s \cong 0$.

Note that in the above treatment for rivers the formulation for momentum balance is decoupled from that for the balance of suspended sediment. The formulation for momentum, sand and mud balance can be solved to yield the results

$$u_o = u_s + gS_s\int_0^z \frac{h_o - z'}{\varepsilon(z')}dz'$$

$$c_{so} = E\exp\left(-v_s\int_0^z \frac{dz'}{\varepsilon(z')}\right) \tag{10a,b,c}$$

$$c_{mo} = c_{moo}$$

where c_{moo} is a specified constant. The above forms can in turn be written as

$$\frac{u_o}{\overline{u}_o} = f_{ru}(\varsigma) \qquad \frac{c_{so}}{\overline{c}_{so}} = f_{rs}(\varsigma) \qquad \frac{c_{mo}}{\overline{c}_{mo}} = f_{rm}(\varsigma) \tag{11a,b,c}$$

where

$$\varsigma = \frac{z}{h_o} \tag{12}$$

$$\overline{u}_o = \frac{1}{h_o}\int_0^{h_o} u_o dz \quad \overline{c}_{so} = \frac{1}{h_o}\int_0^{h_o} c_s dz \quad \overline{c}_{mo} = \frac{1}{h_o}\int_0^{h_o} c_{mo} dz = c_{moo} \tag{13a,b,c}$$

and

$$f_{ru} = \frac{u_s + gS_s h_o^2 \int_0^{\varsigma} \frac{1-\varsigma'}{\varepsilon} d\varsigma'}{u_s + gS_s h_o^2 \int_0^1 \int_0^{\varsigma} \frac{1-\varsigma'}{\varepsilon} d\varsigma' d\varsigma}$$

$$f_{rs} = \frac{\exp\left(-v_s h_o \int_0^{\varsigma} \frac{d\varsigma'}{\varepsilon}\right)}{\int_0^1 \exp\left(-v_s h_o \int_0^{\varsigma} \frac{d\varsigma'}{\varepsilon}\right) d\varsigma} \tag{14a,b,c}$$

$$f_{rm} = 1$$

Based on the above formulation, the Froude number **Fr** of the river flow is defined as

$$\mathbf{Fr} = \frac{\overline{u}_o}{\sqrt{gh_o}} \tag{15}$$

The volume transport rate of suspended mud and sand per unit width in the streamwise direction, q_{smo} and q_{sso}, respectively, are given as

$$q_{smo} = \int_0^{h_o} u_o c_{mo} dz = \overline{u}_o \overline{c}_{mo} h_o \int_0^1 f_{ru}(\varsigma) f_{rm}(\varsigma) d\varsigma$$
$$q_{sso} = \int_0^{h_o} u_o c_{so} dz = \overline{u}_o \overline{c}_{so} h_o \int_0^1 f_{ru}(\varsigma) f_{rs}(\varsigma) d\varsigma \tag{16a,b}$$

Sand may also be transported as bedload. The volume bedload transport rate per unit width q_{sbo} is typically linked to sand grain size D_s and boundary shear stress τ_{sbo}. The ratio of suspended sand load to bedload is here denoted as φ, where

$$\varphi = \frac{q_{sso}}{q_{sbo}} \tag{17}$$

A specific example of an implementation of the above scheme for open channel flow is given in e.g. Parker and Johannesson (1989), which uses the simple algebraic turbulence closure scheme due to Engelund (1974). In that analysis ε is approximated as a constant given by the relation

$$\varepsilon = \frac{1}{13} u_{*o} h_o \tag{18}$$

In addition the bottom boundary condition (7b) takes the specific form

$$u_s = u_o\big|_{z=0} = \chi h_o \frac{du_o}{dz}\bigg|_{z=0} \tag{19a}$$

where

$$\chi = \chi_1 - \frac{1}{3} \qquad \chi_1 = \frac{C_f^{-1/2}}{13} \qquad C_f = \left(\frac{u_{*o}}{\bar{u}_o}\right)^2 \tag{19b,c,d}$$

Here χ and χ_1 are order-one parameters, the former varying between 0.44 and 1.97 and the latter varying between 0.77 and 2.31 as $C_f^{-1/2}$ varies between 10 and 30. The above assumptions specify the following forms for f_{ru} and f_{rs}:

$$f_{ru} = \frac{\chi + \varsigma - \frac{1}{2}\varsigma^2}{\chi_1} \qquad f_{rs} = \frac{v_s h_o}{\varepsilon} \frac{\exp(-\frac{v_s h_o}{\varepsilon}\varsigma)}{1 - \exp(-\frac{v_s h_o}{\varepsilon})} \tag{20a,b}$$

In addition, the relation between depth-averaged sand concentration $\bar{c}_{so}$ and the dimensionless sand entrainment parameter E is found from (13b) to be

$$\bar{c}_{so} = \frac{\varepsilon}{v_s h_o}\left[1 - \exp\left(-\frac{v_s h_o}{\varepsilon}\right)\right] E \tag{21}$$

Garcia and Parker (1991) provide examples of functional forms for E as a function of boundary shear stress and other parameters.

3.2 Channelized turbidity current

As noted above, the turbidity current is assumed to be subcritical in the bulk Richardson sense, and to have sufficiently strong stratification in the vicinity of the interface between turbid and clear water so as to allow for a neglect of interfacial mixing at first order. The flow thickness z = h_o of the turbidity current is defined to be the point above the velocity maximum at which the magnitude of the shear stress τ_{io} takes its maximum value, as shown in Figure 4. Here τ_{io} is referred to as the interfacial shear stress, by definition given by the relation

$$-\varepsilon \left.\frac{du_o}{dz}\right|_{z=h_o} = \frac{1}{\rho}\tau_{io} \qquad (22)$$

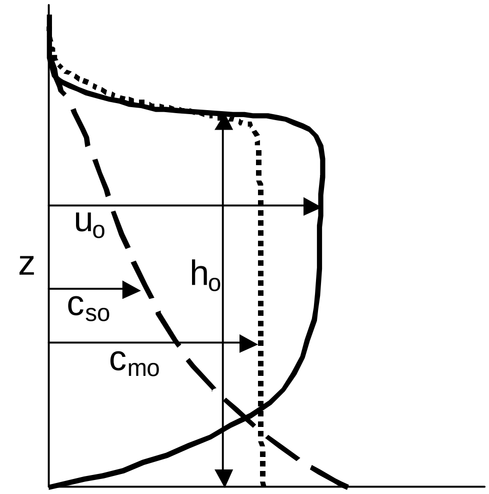

Figure 4. Definition diagram for the vertical structure of a turbidity current.

A condition for the interface to be relatively sharp is a sufficiently high gradient Richardson number **Ri** there, i.e.

$$\mathbf{Ri} = -\left.\frac{Rg\dfrac{d(c_{mo}+c_{so})}{dz}}{\left(\dfrac{du_o}{dz}\right)^2}\right|_{z=h_o} \qquad (23)$$

Here a formulation for turbidity currents that is valid over the range $0 \le z \le h_o$ is sought. The turbidity current above z = h_o is neglected in accordance with the assumption of a sharp interface illustrated in Figure 4.

The formulation of momentum balance of (7a-c) for rivers takes the following form for turbidity currents:

$$0 = RgS_s(c_{mo}+c_{so}) + \frac{d}{dz}\varepsilon(z)\frac{du_o}{dz} \qquad (24a)$$

and

$$u_o(0) = u_s \qquad -\varepsilon\left.\frac{du_o}{dz}\right|_{z=h_o} = \tau_{io} \qquad (24b,c)$$

The formulations for mass balance of sand and mud remain unaltered from their forms for rivers. A reduction of the relations involved leads to the following solutions: u_o is given by the relation below,

$$u_o = h_o \int_0^{\varsigma} \frac{1}{\varepsilon}\left[R g S_s h_o \int_{\varsigma'}^{1} (\overline{c}_{mo} + \overline{c}_{so}) d\varsigma'' - \frac{\tau_{io}}{\rho}\right] d\varsigma' \tag{25}$$

and c_{mo} and c_{so} are again given by the relations (10b) and (10c) respectively. In addition,

$$\frac{u_o}{\overline{u}_o} = f_{tu}(\varsigma) \qquad \frac{c_{so}}{\overline{c}_{so}} = f_{ts}(\varsigma) \qquad \frac{c_{mo}}{\overline{c}_{mo}} = f_{tm}(\varsigma) \tag{26a,b,c}$$

where $\overline{u}_o$, $\overline{c}_{mo}$ and $\overline{c}_{so}$ are again given by (13), and

$$f_{tu} = \frac{\int_0^{\varsigma} \frac{1}{\varepsilon}\left[R g S_s h_o \int_{\varsigma'}^{1} (\overline{c}_{mo} + \overline{c}_{so}) d\varsigma'' - \frac{\tau_{io}}{\rho}\right] d\varsigma'}{\int_0^1 \int_0^{\varsigma} \frac{1}{\varepsilon}\left[R g S_s h_o \int_{\varsigma'}^{1} (\overline{c}_{mo} + \overline{c}_{so}) d\varsigma'' - \frac{\tau_{io}}{\rho}\right] d\varsigma' d\varsigma}$$

$$f_{ts} = \frac{\exp\left(- v_s h_o \int_0^{\varsigma} \frac{d\varsigma'}{\varepsilon}\right)}{\int_0^1 \exp\left(- v_s h_o \int_0^{\varsigma} \frac{d\varsigma'}{\varepsilon}\right) d\varsigma} \tag{27a,b,c}$$

$$f_{tm} = 1$$

Note from the form of (24a) that in the case of a turbidity current the velocity profile is strongly coupled to the concentration profile. Although not explicit in the present formulation, even algebraic closures for ε link the concentration profile to the velocity profile, as was seen for the Engelund (1974) closure applied above to rivers..

The above formulation allows the definition of a densimetric Froude number $\mathbf{Fr}_d$ and a bulk Richardson number $\mathbf{Ri}_b$ for the primary flow:

$$\mathbf{Fr}_b = \frac{\overline{u}_o}{\sqrt{R g(\overline{c}_{mo} + \overline{c}_{so}) h_o}} \qquad \mathbf{Ri}_b = \frac{R g(\overline{c}_{mo} + \overline{c}_{so}) h_o}{\overline{u}_o^2} \tag{28a,b}$$

The condition for a subcritical turbidity current is $\mathbf{Fr}_d < 1$ or equivalently $\mathbf{Ri}_b > 1$.

4. SECONDARY FLOW

Here the case of secondary flow subject to the conditions (2a,b) is considered. The transverse bed slope S_{tb} is assumed to be positive when bed elevation η decreases in the positive n direction. The transverse water surface or interfacial slope S_{tt} is assumed to be positive when the elevation of the water surface or interface ξ increases in the positive n direction. Noting that

$$\xi \cong \eta + h \tag{29a}$$

it follows that

$$S_{tb} = -\frac{\partial \eta}{\partial n} \qquad S_{tt} = \frac{\partial \xi}{\partial n} = -S_{tb} + \frac{dh}{dn} \tag{29b,c}$$

These sign conventions are illustrated in Figure 2. The secondary flow itself is denoted as v(z), and is evaluated everywhere except in thin layers near the sidewalls.

4.1 River

The equation of conservation of transverse momentum balance can be approximated by the following linearized form in the case of uniform flow in a bend satisfying (2a):

$$-\frac{u_o^2}{r_o} = -\frac{1}{\rho}\frac{dp}{dn} + gS_{tb} + \frac{d}{dz}\left(\varepsilon \frac{dv}{dz}\right) \tag{30}$$

Here the pressure distribution can be approximated as hydrostatic, such that

$$p = \rho g(h - z) \tag{31a}$$

or thus

$$-\frac{dp}{dn} = -\rho g \frac{dh}{dn} = -\rho g(S_{tb} + S_{tt}) \tag{31b}$$

Substituting (31b) into (30), it is found that

$$-\frac{u_o^2}{r_o} = -gS_{tt} + \frac{d}{dz}\left(\varepsilon \frac{dv}{dz}\right) \tag{32a}$$

The boundary conditions on (32a) can be written as

$$v(0) = v_s \qquad \varepsilon \left.\frac{dv}{dz}\right|_{z=h_0} = 0 \tag{32b,c}$$

where v_s is a slip velocity analogous to the slip velocity u_s of the primary flow. In addition, conservation of flow mass requires the integral constraint

$$\int_0^{h_o} v\,dz = 0 \tag{32d}$$

The transverse boundary shear stress at the bed is given by the relation

$$\varepsilon \left.\frac{dv}{dz}\right|_{z=0} = \frac{\tau_{nb}}{\rho} \tag{32e}$$

The problem can be reformulated in dimensionless terms as follows;

$$u_0^2 = \bar{u}_0^2 f_{ru}^2(\varsigma) \qquad \varepsilon = \bar{u}_o h_o \hat{\varepsilon} \qquad \delta = \frac{h_o}{r_o}$$
$$S_{tt} = \delta \hat{S}_{tt} \qquad v = \delta \bar{u}_o \hat{v} \tag{33a-e}$$

according to which (32a-d) can be rewritten as

$$-f_{ru}^2(\varsigma) = -\mathbf{Fr}^{-2}\hat{S}_{tt} + \frac{d}{d\varsigma}\left(\hat{\varepsilon}\frac{d\hat{v}}{d\varsigma}\right)$$
$$\hat{v}(0) = \hat{v}_s \qquad \hat{\varepsilon}\left.\frac{d\hat{v}}{d\varsigma}\right|_{\varsigma=1} = 0 \qquad \int_0^1 \hat{v}\,d\varsigma = 0 \tag{34a-d}$$

Equations (34a-d) represent a single second-order relation subject to three boundary conditions. As a result it is possible to integrate to determine solutions to both $\hat{v}$ and the constant $\hat{S}_{tt}$. The results are found to be

$$\hat{v}(\varsigma) = \hat{v}_s + \int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\left[\int_{\varsigma'}^1 f_{ru}^2(\varsigma'')d\varsigma'' - \alpha_{rtt}(1-\varsigma')\right]d\varsigma'$$
$$\hat{S}_{tt} = \alpha_{rtt}\mathbf{Fr}^2 \tag{35a,b}$$

where

$$\alpha_{rtt} = \frac{\hat{v}_s + \int_0^1\int_0^{\varsigma}\frac{1}{\hat{\varepsilon}(\varsigma')}\int_{\varsigma'}^1 f_{ru}^2(\varsigma'')d\varsigma''d\varsigma'd\varsigma}{\int_0^1\int_0^{\varsigma}\frac{1}{\hat{\varepsilon}(\varsigma')}(1-\varsigma')d\varsigma'd\varsigma} \tag{35c}$$

As an example, the above scheme is implemented using the Engelund (1974) closure procedure described above. In that approach the slip condition (34b) takes the specific form

$$\hat{v}_{ss} = \hat{v}\big|_{\varsigma=0} = \chi\left.\frac{d\hat{v}}{d\varsigma}\right|_{\varsigma=0} \tag{36}$$

Johannesson and Parker (1989) give the solution for α_{rtt} as

$$\alpha_{rtt} = \frac{\left(\chi^3 + \chi^2 + \frac{2}{5}\chi + \frac{2}{35}\right)}{\left(\chi + \frac{1}{3}\right)^3} \tag{37}$$

In addition, they find the secondary flow to obey the relation

$$\hat{v} = \frac{G_o(\varsigma)}{C_f \chi_1} \tag{38}$$

where

$$G_o = \frac{1}{\chi_1^2}\left[\left(\chi^2 + \frac{2}{3}\chi + \frac{2}{15}\right)(\chi + \varsigma) - \frac{1}{2}\chi^2\varsigma^2 - \frac{1}{3}\chi\varsigma^3 - \frac{1}{12}(1-\chi)\varsigma^4 + \frac{1}{20}\varsigma^5 - \frac{1}{120}\varsigma^6\right] - \alpha_{rtt}\left(\chi + \varsigma - \frac{1}{2}\varsigma^2\right) \tag{39}$$

The above relations serve to verify the form of (1a) for superelevation at a river bend. Between (33d), (35b) and (37) the following result is obtained:

$$S_{tt} = \alpha_{rtt}\frac{h_o}{r_o}\mathbf{Fr}^2 \tag{40}$$

Referring to (19b,c) it is found that α_{rtt} varies from 1.11 to 1.01 as $C_f^{-1/2}$ varies from 10 to 30. Thus (40) is identical to (1a) up to a coefficient that varies only modestly from unity over a broad range of conditions.

4.2 Channelized turbidity current

The equation of transverse momentum balance for a turbidity current corresponding to (24) can be written as

$$-\frac{u_o^2}{r_o} = -\frac{1}{\rho}\frac{dp_e}{dn} + Rg(c_{mo} + c_{so})S_{tb} + \frac{d}{dz}\left(\varepsilon\frac{dv}{dz}\right) \tag{41}$$

where p_e is an extra component of hydrostatic pressure associated with the presence of sediment, and satisfying the relations

$$\frac{dp_e}{dz} = -Rg(c_{mo} + c_{so}) \tag{42a}$$

or thus

$$p_e = Rg\int_z^{h_o}(c_{mo} + c_{so})dz \tag{42b}$$

The boundary conditions on (41) are

$$v(0) = v_s \qquad -\varepsilon\frac{dv}{dz}\bigg|_{z=h_0} = \frac{\tau_{ni}}{\rho} \qquad \int_0^{h_o} vdz = 0 \tag{43a,b,c}$$

where τ_{ni} denotes the transverse interfacial shear stress, and the transverse bed shear stress τ_{nb} is again given by (32e).

Using (12), (26b,c), (29b,c) and (42a) to reduce the pressure term in (41), it is found that

$$\begin{aligned} \frac{1}{\rho}\frac{dp_e}{dn} &= Rg\frac{\partial}{\partial n}\{h(\bar{c}_{mo} + \bar{c}_{so})\int_\varsigma^1[f_{tm}(\varsigma') + f_{ts}(\varsigma')]d\varsigma'\} \\ &= Rg(\bar{c}_{mo} + \bar{c}_{so})(S_{tt} + S_{tb})\left\{\varsigma[f_{tm}(\varsigma) + f_{ts}(\varsigma)] + \int_\varsigma^1[f_{tm}(\varsigma') + f_{ts}(\varsigma')]d\varsigma'\right\} \end{aligned} \tag{44}$$

Using (44) to reduce (41) and rendering (41) and (43) dimensionless according to (26b,c), (33a-e) and the relations

$$S_{tb} = \delta\hat{S}_{tb} \qquad \tau_{ni} = \rho\bar{u}_o^2\delta\hat{\tau}_{ni} \tag{45a,b}$$

the following dimensionless formulation for turbidity currents is obtained:

$$-f_{tu}^2(\varsigma) = -\mathbf{Fr}_d^{-2}(\hat{S}_{tt}X_t + \hat{S}_{tb}X_b) + \frac{d}{d\varsigma}\left(\hat{\varepsilon}\frac{d\hat{v}}{d\varsigma}\right) \tag{46a}$$

where

$$\begin{aligned} X_t &= \varsigma[f_{tm}(\varsigma) + f_{ts}(\varsigma)] + \int_\varsigma^1[f_{tm}(\varsigma') + f_{ts}(\varsigma')]d\varsigma' \\ X_b &= -(1-\varsigma)[f_{tm}(\varsigma) + f_{ts}(\varsigma)] + \int_\varsigma^1[f_{tm}(\varsigma') + f_{ts}(\varsigma')]d\varsigma' \end{aligned} \tag{46b,c}$$

and

$$\hat{v}(0) = \hat{v}_s \qquad -\hat{\varepsilon}\frac{d\hat{v}}{d\varsigma}\bigg|_{\varsigma=1} = \hat{\tau}_{ni} \qquad \int_0^1 \hat{v}\,d\varsigma = 0 \tag{46d,e,f}$$

The solution to (46) is found to be

$$\begin{aligned}\hat{v} = \hat{v}_s + \int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\{-\hat{\tau}_{ni} + \int_{\varsigma'}^1 f_{tu}^2(\varsigma'')d\varsigma'' - \mathbf{Fr}_d^{-2}[\hat{S}_{tt}\int_{\varsigma'}^1 X_t(\varsigma'')d\varsigma'' \\ + \hat{S}_{tb}\int_{\varsigma'}^1 X_b(\varsigma'')d\varsigma'']\}d\varsigma'\end{aligned} \tag{47a}$$

where

$$\gamma_{tt}\hat{S}_{tt} + \gamma_{tb}\hat{S}_{tb} = \gamma\mathbf{Fr}_d^2 \tag{47b}$$

and

$$\begin{aligned}\gamma_{tt} &= \int_0^1\int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\int_{\varsigma'}^1 X_t(\varsigma'')d\varsigma''d\varsigma'd\varsigma \\ \gamma_{tb} &= \int_0^1\int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\int_{\varsigma'}^1 X_b(\varsigma'')d\varsigma''d\varsigma'd\varsigma \\ \gamma &= \int_0^1\int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\left[-\hat{\tau}_{ni} + \int_{\varsigma'}^1 f_{tu}^2(\varsigma'')d\varsigma''\right]d\varsigma'd\varsigma + \hat{v}_s\end{aligned} \tag{47c,d,e}$$

Equation (47b) contains one of the two important results of the present analysis. In a subcritical turbidity current capable of forming a meandering channel, mud is likely to be distributed relatively uniformly in the vertical, whereas sand, which is much heavier, is likely to be characterized by a fairly strong concentration gradient in the vertical. It is this upward gradient of sand concentration that renders turbidity currents fundamentally different from rivers. If for example the current is assumed to contain no sand, so that f_{ts} can be dropped in (46b,c), and f_{tm} is evaluated using (27c) it is found that

$$X_b(\varsigma) = \gamma_{tb} = 0 \tag{48}$$

so that the (47) reduces to

$$\begin{aligned}\hat{v}(\varsigma) &= \hat{v}_s + \int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\left[\int_{\varsigma'}^1 f_{tu}^2(\varsigma'')d\varsigma'' - \alpha_{ttt}(1-\varsigma')\right]d\varsigma' \\ \hat{S}_{tt} &= \alpha_{ttt}\mathbf{Fr}_d^2\end{aligned} \tag{49a,b}$$

where

$$\alpha_{ttt} = \frac{\gamma}{\gamma_{tt}} = \frac{\hat{v}_s + \int_0^1 \int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')} \left[-\hat{\tau}_{ni} + \int_{\varsigma'}^1 f_{tu}^2(\varsigma'')d\varsigma'' \right] d\varsigma' d\varsigma}{\int_0^1 \int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}(1-\varsigma')d\varsigma' d\varsigma} \tag{49c}$$

That is, the structure of the "solution" is identical to that for rivers save the generalization to densimetric Froude number and the added transverse interfacial stress term. Equation (49b) for superelevation reduces to the dimensioned form

$$S_{tt} = \alpha_{ttt} \frac{h_o}{r_o} \mathbf{Fr}_d^2 \tag{50}$$

While the present analysis does not allow for the evaluation of the constant α_{ttt}, it is likely order-one and sufficiently close to unity to allow for a computation of bend superelevation of muddy subcritical turbidity currents from (50) with α_{ttt} approximated to unity.

The case in which sand is a significant component of the turbidity current, however, offers a very different structure. In particular, it is seen from (47b) that the transverse surface slope is coupled to the transverse bed slope. As a result it is not possible to compute superelevation around bends in the absence of a predictive formulation for transverse bed slope.

5. TRANSVERSE SEDIMENT TRANSPORT AND BED SLOPE

As noted above, it is assumed here that the mud in either a river or a turbidity current is not contained in, eroded from or deposited on the sand bed. For uniform bend flow the Exner equation of conservation of bed sand can be written in the form

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = v_s[c_o(0) - E] - \frac{\partial q_{nb}}{\partial n} \tag{51a}$$

Here the first term on the right-hand side of (51) denotes the net rate of settling of sand on the bed from suspension and the second term denotes the divergence of the transverse volume bedload transport rate per unit width q_{nb}. For steady flow (51a) reduces to

$$0 = v_s[c_{so}(0) - E] - \frac{dq_{nb}}{dn} \tag{51b}$$

The equation of conservation of suspended sediment in the water column for steady flow that is uniform in the streamwise direction is

$$\frac{dq_{ns}}{dn} = v_s[E - c_{so}(0)] \tag{52a}$$

where q_{ns} denotes the transverse volume transport rate of suspended sand per unit width, given by

$$q_{ns} = \int_0^{h_o} v c_{so} dz \qquad (52b)$$

Between (51) and (52),

$$\frac{d}{dn}(q_{nb} + q_{ns}) = 0 \qquad (53)$$

Integrating (53) under the condition of vanishing transverse bedload and suspended sand load at the sidewalls, it is found that

$$q_{nb} + q_{ns} = 0 \qquad (54)$$

The above equation is a necessary condition for steady, uniform bend flow over a sand bed of a single grain size.

In considering transverse bedload transport in a bend, it is important to realize that the sand will have an inherent tendency to roll down any transverse slope. A variety of relations have been developed to predict this tendency (e.g. Parker and Johannesson, 1989). In line with the linearized analysis presented above, the formulation of Parker and Andrews (1985) is used here. When applied to steady, uniform bend flow it takes the form

$$\frac{q_{nb}}{q_{sbo}} = \frac{\tau_{nb}}{\tau_{sbo}} + \beta S_{tb} \qquad (55a)$$

where

$$\beta = \beta^* \sqrt{\frac{\tau_c^*}{\tau_{so}^*}} \qquad \tau_{so}^* = \frac{\tau_{sbo}}{\rho R g D_s} \qquad (55b)$$

In the above relation β^* is an order-one constant and τ_c^* denotes a critical Shields stress for the onset of motion. In general the solution to secondary flow in bends at linear level yields a solution for τ_{nb} of the form

$$\frac{\tau_{nb}}{\tau_{sbo}} = -\delta a_b \qquad (56a)$$

where a_b is an order-one parameter. For example, in Parker and Johannesson's (1989) implementation of the Engelund (1974) closure it is found that

$$a_b = 7.51 \frac{\chi + \frac{2}{7}}{\chi_1} \tag{56b}$$

The transverse sand transport rate q_{ns} can be similarly related to the streamwise sand transport rate q_{sso} as follows:

$$q_{ns} = \frac{q_{ns}}{q_{sso}} q_{sso} = -\delta a_s q_{sso} \tag{57}$$

where according to (16b), (33e) and (52b)

$$a_s = -\frac{\int_0^1 \hat{v}(\varsigma) f_{rs}(\varsigma) d\varsigma}{\int_0^1 f_{ru}(\varsigma) f_{rs}(\varsigma) d\varsigma} \tag{58a}$$

for a river and

$$a_s = -\frac{\int_0^1 \hat{v}(\varsigma) f_{ts}(\varsigma) d\varsigma}{\int_0^1 f_{tu}(\varsigma) f_{ts}(\varsigma) d\varsigma} \tag{58b}$$

for a turbidity current. The minus sign on the right-hand side of (58a) and (58b) insures that a_s is positive for the physically realistic case of a vertical distribution of sand concentration that is higher toward the bed, where the secondary flow is directed inward (i.e. in the negative n direction).

Substituting (55a), (56a) and (57) into (54) and reducing with (45a), it is found that

$$\hat{S}_{tb} = \frac{1}{\beta}(a_b + \varphi a_s) \tag{59}$$

where φ is the ratio of streamwise suspended transport rate to streamwise bedload rate, as defined by (17).

5.1 River

In the case of a river carrying sand only as bedload, (59) reduces to

$$\hat{S}_{tb} = \frac{a_b}{\beta} \equiv A_b \tag{60a}$$

or in dimensioned terms,

$$S_{tb} = A_b \frac{h_o}{r_o} \tag{60b}$$

The formulation of (60) is the one inherent in Parker and Johannesson (1989). When sand is also carried as suspended load, (60a,b) generalize to the respective forms

$$\hat{S}_{tb} = \frac{a_b + \varphi a_s}{\beta} \equiv A$$
$$S_{tb} = A \frac{h_o}{r_o} \tag{61a,b}$$

The formulations of (60) and (61) guarantee that the transverse bed slope is decoupled from the transverse water surface slope. In particular, according to (55b) and (56b) a_b and β are independent of S_{tt}. In addition, the forms of (35a) and (35c) are such that a_s must also be independent of S_{tt}.

To summarize, in a river the relation (40) provides a predictor of water surface slope S_{tt} (superelevation) and (61b) provides a predictor of bed slope S_{tb}. The two relations are decoupled from each other.

5.2 Channelized turbidity current

Equations (61a,b) also determine the transverse bed slope of a channelized turbidity current. This bed slope is no longer, however independent of the transvese interfacial slope. This can be seen by substituting (47a) into (58b) and reducing to obtain

$$a_s = a_{so} - \mathbf{Fr}_d^{-2}(\hat{S}_{tt} a_{st} + \hat{S}_{tb} a_{sb}) \tag{62a}$$

where

$$a_{so} = -\frac{\int_0^1 f_{ts}(\varsigma)\int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\left[-\hat{\tau}_{ni} + \int_{\varsigma'}^1 f_{tu}^2(\varsigma'')d\varsigma''\right]d\varsigma' d\varsigma}{\int_0^1 f_{tu}(\varsigma) f_{ts}(\varsigma) d\varsigma} \tag{62b}$$

$$a_{st} = -\frac{\int_0^1 f_{ts}(\varsigma)\int_0^{\varsigma} \frac{1}{\hat{\varepsilon}(\varsigma')}\int_{\varsigma'}^1 X_t(\varsigma'')d\varsigma'' d\varsigma' d\varsigma}{\int_0^1 f_{tu}(\varsigma) f_{ts}(\varsigma) d\varsigma} \tag{62c}$$

and

$$a_{sb} = -\frac{\int_0^1 f_{ts}(\varsigma)\int_0^{\varsigma}\frac{1}{\hat{\varepsilon}(\varsigma')}\int_{\varsigma'}^1 X_b(\varsigma'')d\varsigma''d\varsigma'd\varsigma}{\int_0^1 f_{tu}(\varsigma)f_{ts}(\varsigma)d\varsigma} \tag{62d}$$

Substituting (62a) into (61a), it is found that

$$\hat{S}_{tb} = \frac{a_b + \varphi[a_{so} - \mathbf{Fr}_d^{-2}(a_{st}\hat{S}_{tt} + a_{sb}\hat{S}_{tb})]}{\beta} \tag{63}$$

Equations (47b) and (63) define two coupled equations for the two unknowns $\hat{S}_{tt}$ and $\hat{S}_{tb}$. They can be solved to yield

$$\hat{S}_{tt} = \frac{\gamma}{\gamma_{tt}}\mathbf{Fr}_d^2 - \frac{\gamma_{tb}}{\gamma_{tt}}\frac{a_b + \varphi\left(a_{so} - a_{st}\frac{\gamma}{\gamma_{tt}}\right)}{\beta + \varphi\mathbf{Fr}_d^{-2}\left(a_{sb} + a_{st}\frac{\gamma_{tb}}{\gamma_{tt}}\right)}$$

$$\hat{S}_{tb} = \frac{a_b + \varphi\left(a_{so} - a_{st}\frac{\gamma}{\gamma_{tt}}\right)}{\beta + \varphi\mathbf{Fr}_d^{-2}\left(a_{sb} + a_{st}\frac{\gamma_{tb}}{\gamma_{tt}}\right)} \tag{64a,b}$$

A comparison of (64a,b) for the case of turbidity currents with (40) and (61a) for the case of rivers reveals an essentially different structure between the two. It thus becomes clear that (1a,b,c) are insufficient as descriptors of transverse interfacial and bed slopes in the case of subcritical turbidity currents with a sharp interface carrying substantial quantities of sand as well as mud. The processes by which transverse bed and water surface slopes are maintained for such flows are strongly coupled.

6. CONCLUSION

Strongly meandering channels on the bottom of the ocean extending for hundreds or thousands of kilometers suggest that channelized submarine turbidity currents behave in similar ways to alluvial rivers. Here the problem of steady, streamwise-uniform flow in a channel of mild, constant curvature is investigated for parallel cases of a river and a channelized turbidity current. In both cases the flow carries both sand and mud. The bed of the channel is composed of uniform sand, and this sand is carried as bedload and suspended load. The mud is transported as wash load.

Interest is focused on Froude-subcritical flows, in the standard sense for a river and in the densimetric sense for a channelized turbidity current. In the case of the turbidity current the flow is assumed to be sufficiently strongly stratified so as to have a relatively sharp interface between turbid and clear water at the top of the flow. Such an interface suppresses the entrainment of clear water from above, so allowing the flow to follow the channel for long distances.

The analysis presented here is general, but cannot be implemented in any specific case unless a turbulent closure is specified. Standard forms for predictors of transverse water surface and bed slopes are verified for the case of rivers, in which the processes maintaining the respective slopes decouple. In the case of a turbidity current carrying a substantial amount of sand with a strong vertical gradient in sand concentration, however, the standard predictors for rivers cannot be simply generalized with the use of the densimetric Froude number. In particular, the processes that maintain interfacial and bed slopes may be strongly coupled. Precise predictions based on the analysis presented here await the implementation of an appropriate closure scheme.

This research was supported by the National Science Foundation (grant no. NSF-OCE-9711431) and the STRATAFORM program of the Office of Naval Research (grant no. N/N00014-93-1-0300).

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