LECTURE NOTES ON THE INTERACTION OF FLOW AND SEDIMENT TRANSPORT IN RIVERS AND THE OCEAN

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ABSTRACT

These lecture notes provide an introduction into the mechanics of flow-sediment interaction in rivers and the ocean, as well as a description of the resulting morphologies. In the first chapter the 1- and 2-dimensional de St. Venant equations of shallow water flow are derived from the Navier-Stokes equations and the boundary layer approximations. In the second chapter relations are developed for the description of sediment mass balance via the Exner equation of sediment continuity. In the third chapter a description is given of some simple relations for bedload and suspended load transport. In the fourth chapter the mechanisms for dunes, antidunes, alternate bars, meandering and the formation of large scale morphology are presented. The fifth chapter provides an introduction to oceanic turbidity currents. The sixth chapter considers applications of the above material to applied problems.

CHAPTER 1
DERIVATION OF THE SHALLOW WATER EQUATIONS

1.1 INTRODUCTION

The interaction of flow and sediment transport in rivers creates a variety of interesting phenomena and morphologies, including dunes, bars, meandering, alluvial fans, submarine channels etc. In most cases of interest the flow can be expected to be fully turbulent. The tools used to model the flow and sediment transport depend upon the scales of interest. At the scale of dunes and ripples, the mechanics of sediment transport must be coupled with the Reynolds-averaged Navier Stokes equations (appropriately closed for turbulence) to describe the phenomenon. At larger scales, however, the shallow-water equations are quite adequate to model the flow. Since such scales are commonly the focus of sediment transport and river mechanics, a derivation of the de St. Venant equations from first principles is provided here.

1.2 REYNOLDS EQUATIONS

The fluid in question in rivers and the ocean is water. In this chapter the water is assumed to be incompressible and of constant density. Let u_i denote the instantaneous velocity vector of the water flow and p denote the instantaneous pressure. The Navier-Stokes equations for an incompressible fluid can be written as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + g_i \tag{1.1}$$

In the above relation, t denotes time x_i denotes the spatial vector, ρ denotes the density of the fluid, ν denotes the kinematic viscosity of the fluid and g_i denotes the vector of gravitational acceleration, given by

$$g_i = -g\, n_{vi} \tag{1.2}$$

where g denotes the magnitude of gravitational acceleration and n_{vi} is a unit upward vertical vector. The relation for water continuity is

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1.3}$$

The flow may be averaged over the turbulence using a standard Reynolds decomposition:

$$u_i = \overline{u}_i + u'_i \qquad p = \overline{p} + p' \tag{1.4}$$

where the overbar denotes the averaging and the prime denotes fluctuations about the average. Averaging results in the following Reynolds forms for water momentum and mass conservation:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial \overline{\tau}_{vij}}{\partial x_j} + \frac{1}{\rho}\frac{\partial \tau_{Rij}}{\partial x_j} + g_i \tag{1.5}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{1.6}$$

where

$$\overline{\tau}_{vij} = \rho\nu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \qquad \tau_{Rij} = -\rho\overline{u'_i u'_j} \tag{1.7}$$

denote the average viscous stress tensor and the Reynolds stress tensor due to turbulence, respectively.

1.3 APPLICATION TO A RIVER

The above equations are applied to free-surface flow in a river. For simplicity the flow is considered two-dimensional, with a streamwise and normal coordinate. Boundary layer coordinates are used, so that x denotes the streamwise direction **along the bottom boundary of the river** and z denotes an upward coordinate **normal to the bottom of the river**. The river bottom has been assumed to be averaged over small-scale roughness elements such as ripples and dunes. The flow is considered to be fully turbulent except possibly for a thin viscous layer near the bed. In the great majority of cases the turbulence can be considered to be hydraulically rough, so even the viscous layer can be neglected.

The geometry is illustrated in the figure below. The flow depth H(x, t) is measured normal to the bed. Vertical bed elevation is denoted by $\eta(x,t)$; bed slope S is given by

$$S = -\frac{\partial \eta}{\partial x} \tag{1.8}$$

For most rivers S is quite small, varying from $1x10^{-5}$ to $4x10^{-2}$. Under these conditions vectorial gravitational acceleration is accurately approximated by

$$g_i = g(S, \quad -1) \tag{1.9}$$

where the components of the vector are in the x and z directions, in that order.

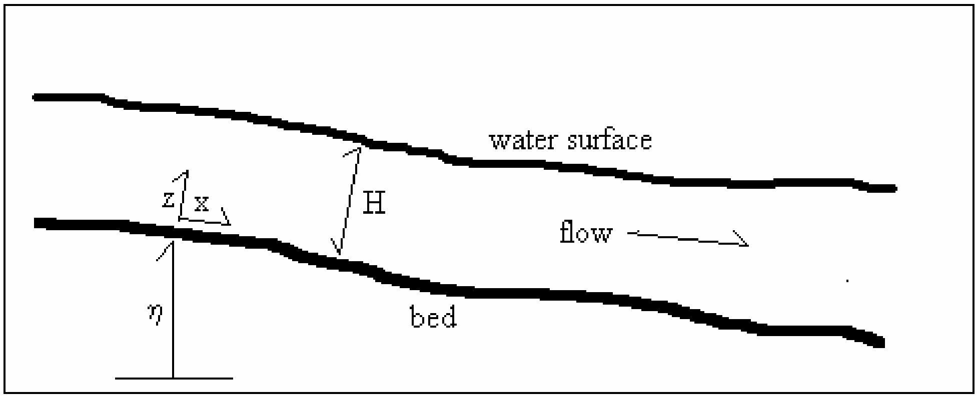

Figure 1.1 Definition diagram.

The key assumption in the boundary layer, or slender flow approximations is that the characteristic length L_x for change in the streamwise direction should be large compared to the depth of flow H;

$$\frac{H}{L_x} \sim \frac{\partial/\partial x}{\partial/\partial z} << 1 \tag{1.10}$$

That is, the flow must change much more slowly in the streamwise direction than in the upward normal direction. For example, consider the flow influenced by a dam downstream. Velocity changes from 0 at z = 0 to its maximum value at z = H, but one must typically go upstream many tens of kilometers to see the slow velocity induced by the dam revert to its equilibrium value upstream. In general the slender flow approximation is excellent for application to all but the smallest scales of interest in rivers.

Let $(\overline{u}, \overline{w})$ denote the two-dimensional average velocity vector in the (x, z) direction. The Reynolds equations thus reduce to

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau_{R11}}{\partial x} + \frac{\partial \tau_{R13}}{\partial z}\right) + gS \tag{1.11}$$

$$\frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial x} + \overline{w}\frac{\partial \overline{w}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \frac{1}{\rho}\left(\frac{\partial \tau_{R31}}{\partial x} + \frac{\partial \tau_{R33}}{\partial z}\right) - g \qquad (1.12)$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{w}}{\partial z} = 0 \qquad (1.13)$$

where

$$\tau_{R11} = -\rho\overline{u'u'} \qquad \tau_{R33} = -\rho\overline{w'w'} \qquad \tau_{R13} = \tau_{R31} = -\rho\overline{u'w'} \qquad (1.14)$$

It can be seen from Figure 1.1 that the coordinate system is intrinsically curvilinear in that it is attached to a boundary that can have changing shape in the streamwise direction. The slender flow approximation (1.10) can be used, however, to show that the metric terms associated with this system can be approximated as unity at lowest order, allowing the coordinate system to be treated as Cartesian.

1.4 THE SLENDER FLOW APPROXIMATIONS

Let U_c denote a characteristic velocity in the streamwise direction and W_c denote a characteristic velocity in the upward normal direction. The equation of continuity (1.13)can be scaled as

$$\frac{\partial \overline{u}}{\partial x} \quad + \quad \frac{\partial \overline{w}}{\partial z} = 0$$

$$o\left(\frac{U_c}{L_x}\right) + o\left(\frac{W_c}{H}\right) \sim 0$$

yielding the scale result

$$W_c \sim \frac{H}{L_x} U_c \qquad (1.15)$$

Thus $W_c/U_c \sim H/L_x << 1$. In scaling the momentum equations, it is useful to introduce the shear velocity u_*, which is related to the streamwise component of the shear stress evaluated at the bed τ_b as follows;

$$\tau_b = \tau_{R13}\big|_{z=0} \equiv \rho u_*^2 \qquad (1.16)$$

Turbulence being generally well correlated for flows of boundary layer type, the following scale estimates can be made;

$$\tau_{R11} \sim \tau_{R33} \sim \tau_{R13} \sim \rho u_*^2$$

Again on empirical grounds, the following condition typically holds for turbulent flows of boundary layer type;

$$\frac{u_*^2}{U_c^2} << 1 \tag{1.17}$$

The momentum balance that would prevail in the upward normal direction in (1.12) in the absence of flow would be the relation for hydrostatic pressure $\overline{p}_h$;

$$-\frac{1}{\rho}\frac{\partial \overline{p}_h}{\partial z} - g = 0 \tag{1.18}$$

This equation can be integrated with the condition of vanishing gage pressure at the water surface to yield the result

$$\overline{p}_h = \rho g(H - z) \tag{1.19}$$

where $\overline{p}_h$ corresponds to the deviation from local atmospheric pressure. For a general flow the pressure $\overline{p}$ can be decomposed into hydrostatic and dynamic components;

$$\overline{p} = \overline{p}_h + \overline{p}_d \tag{1.20}$$

The following scale estimates are introduced for time t and dynamic pressure $\overline{p}_d$;

$$t \sim T_c = \frac{L_x}{U_c} \qquad \overline{p}_d \sim \rho U_c^2 \tag{1.21}$$

Between (1.15), (1.18), (1.20) and (1.21) the equation of upward normal momentum balance can be reduced and scaled as follows;

$$\begin{array}{ccccccc} \dfrac{\partial \overline{w}}{\partial t} & + & \overline{u}\dfrac{\partial \overline{w}}{\partial x} & + & \overline{w}\dfrac{\partial \overline{w}}{\partial z} & = & -\dfrac{1}{\rho}\dfrac{\partial \overline{p}_d}{\partial z} + \dfrac{1}{\rho}\left(\dfrac{\partial \tau_{R31}}{\partial x} + \dfrac{\partial \tau_{R33}}{\partial z}\right) \\ o\left(\dfrac{H}{L_x^2}U_c^2\right) & + & o\left(\dfrac{H}{L_x^2}U_c^2\right) & + & o\left(\dfrac{H}{L_x^2}U_c^2\right) & = & o\left(\dfrac{U_c^2}{H}\right) + o\left(\dfrac{u_*^2}{L_x}\right) + o\left(\dfrac{u_*^2}{H}\right) \end{array}$$

Dividing the scale estimates by U_c^2/H, it is found that

$$o(\frac{H^2}{L_x^2}) + o(\frac{H^2}{L_x^2}) + o(\frac{H^2}{L_x^2}) = o(1) + o(\frac{H}{L_x}\frac{u_*^2}{U_c^2}) + o(\frac{u_*^2}{U_c^2})$$

In light of the slender flow approximation (1.10) and the scale relation for boundary layer turbulence (1.17), the above scalings approximate the upward normal momentum equation to

$$\frac{\partial \overline{p}_d}{\partial z} = 0$$

In light of the fact that dynamic pressure should vanish at the water surface in the absence of wind stresses, it is seen that the $\overline{p}_d$ must be approximated as vanishing everywhere, so that $\overline{p}$ can be approximated by the hydrostatic value $\overline{p}_h$ given by (1.19).

Rephrasing, the essential result of the application of the boundary layer approximations to the equation of momentum balance in the upward normal direction is that the pressure field can everywhere be approximated as hydrostatic.

With this in mind, (1.19) can be substituted into the equation of streamwise momentum balance, which can then be subjected to the boundary layer approximations;

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\frac{\partial H}{\partial x} + \frac{1}{\rho}(\frac{\partial \tau_{R11}}{\partial x} + \frac{\partial \tau_{R13}}{\partial z}) + gS$$

$$o(\frac{U_c^2}{L_x}) + o(\frac{U_c^2}{L_x}) + o(\frac{U_c^2}{L_x}) = o(g\frac{H}{L_x}) + o(\frac{u_*^2}{L_x}) + o(\frac{u_*^2}{H}) + o(gS)$$

Dividing through by U_c^2/L_x, the scale estimates reduce to

$$o(1) + o(1) + o(1) = o(\frac{gH}{U_c^2}) + o(\frac{u_*^2}{U_c^2}) + o(\frac{L_x}{H}\frac{u_*^2}{U_c^2}) + o(\frac{gH}{U_c^2}S\frac{L_x}{H})$$

In general the combination gH/U_c^2, which corresponds to the inverse of the square of the Froude number of the flow, is an order one quantity for river flow, as is the combinatioin S (L_x/H). In fact the gravitational term involving the slope S **must** be approximated as o(1) as it drives the flow. One of the Reynolds stress terms is clearly negligible. Were both of them to be negligible, it would be necessary to drop the effect of shear stress completely from the problem, and in so make steady, uniform flows impossible. With this in mind it follows that

$$\frac{L_x}{H}\frac{u_*^2}{U_c^2} \sim o(1) \qquad L_x \sim H\frac{U_c^2}{u_*^2} \tag{1.22}$$

The streamwise momentum equation thus approximates to

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -g\frac{\partial H}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial z} + gS \tag{1.23}$$

where τ is shorthand notation for τ_{R13}, i.e. the only Reynolds stress retained after the boundary layer approximations.

In summary, the slender flow approximations to the 2-D Reynolds formulation for river flow consist of (1.13) for water mass conservation, (1.23) for streamwise water momentum conservation and (1.19) for upward normal water momentum conservation.

1.5 THE DE ST. VENANT SHALLOW WATER EQUATIONS

A depth averaged streamwise flow velocity U(x, t) can be defined as follows;

$$UH = \int_0^H \bar{u}dz \tag{1.24}$$

In general the structure of the streamwise flow velocity in the upward normal direction can be represented in terms of a dimensionless structure function f, where

$$\frac{\bar{u}}{U} = f(\varsigma, x) \qquad \varsigma = \frac{z}{H} \tag{1.25}$$

Approximate similarity in the velocity structure is satisfied if the dependence of f on can be neglected. Such conditions are often satisfied for the slowly varying flows satisfying the boundary layer approximations. For example, for turbulent rough flow f can be approximated as a function of $\zeta^{1/6}$ independently of x. Substituting (1.25) into (1.24), it is seen that

$$\int_0 f\, d\varsigma = 1 \tag{1.26}$$

Equations (1.13) and (1.23) are now integrated in z from the bed to the water surface. The boundary conditions at the bed are

$$\left.\overline{u}\right|_{z=0} = \left.\overline{v}\right|_{z=0} = 0 \tag{1.27}$$

The kinematic boundary condition applies at the water surface;

$$\frac{\partial H}{\partial t} + \left.\overline{u}\right|_{z=H}\frac{\partial H}{\partial x} = \left.\overline{v}\right|_{z=H} \tag{1.28}$$

In the absence of wind stresses, the following dynamic boundary condition applies at the water surface;

$$\left.\tau\right|_{z=H} = 0 \tag{1.29}$$

The integration proceeds using Leibnitz' rule wherever necessary. For example, in (1.13)

$$\int_0^H \frac{\partial \overline{u}}{\partial x} dz = \frac{\partial}{\partial x}\int_0^H \overline{u} dz - \left.\overline{u}\right|_{z=H}\frac{\partial H}{\partial x}$$

The result for (1.13) is

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(UH) = 0 \tag{1.30}$$

The result for (1.23) is

$$\frac{\partial}{\partial t}(UH) + s\frac{\partial}{\partial x}(U^2 H) = -gH\frac{\partial H}{\partial x} + gHS - \frac{\tau_b}{\rho} \tag{1.31}$$

where again τ_b denotes the evaluation of τ at the bed and s denotes an o(1) shape factor given by

$$s = \int_0^1 f^2(\varsigma) d\varsigma \tag{1.32}$$

For fully turbulent river flow the shape factor s is typically found to be sufficiently close to unity that it is approximated as so. This approximation is used here:

$$s \cong 1 \tag{1.33}$$

1.6 THE QUASI-STEADY APPROXIMATION: FRICTION RELATIONS

The feature of interest when flow is linked to sediment transport is that the bottom boundary becomes deformable. That is, the differential transport of sediment changes the shape of the bottom. It is a cardinal rule of river flow, however, that rivers transport orders of magnitude more water than sediment. This allows for the quasi-steady approximation, according to which the flow is approximated as steady while the bed is evolving. The approximation can be justified rigorously by showing that the characteristic time for the bed to respond to a changed flow is orders of magnitude larger than that for the flow to respond to a changed bed. Under these conditions, and assuming s = 1, the shallow water equations reduce to

$$UH = q_w \tag{1.34}$$

$$U\frac{dU}{dx} = -g\frac{dH}{dx} + gS - C_f\frac{U^2}{H} \tag{1.35}$$

In the above equations q_w denotes the water discharge per unit width and C_f denotes a frictioin coefficient defined as

$$\tau_b = \rho C_f U^2 \tag{1.36}$$

The quasi-steady approximation breaks down for critical and supercritical flow in the Froude sense, i.e. for flows for which the Froude number **Fr** ≥ 1, where

$$\mathbf{Fr} = \frac{U}{\sqrt{gH}} \tag{1.37}$$

In order to close the above equations, it is necessary to introduce a friction relation. Recalling that $\tau_b = \rho u_*^2$, the logarithmic velocity profile for fully rough flow can be specified as

$$\frac{\bar{u}}{u_*} = 2.5\,\ell n(30\frac{z}{k_s}) \tag{1.38}$$

where k_s denotes a roughness height. Integrating this equation according to (1.24), it is found that

$$C_f^{-1/2} = \frac{U}{u_*} = 2.5\,\ell n(11\frac{H}{k_s}) \tag{1.38}$$

The above relation is known as Keulegan's law. An accurate power approximation of it is given by the Manning-Strickler relation;

$$C_f^{-1/2} = 8.1\,(\frac{H}{k_s})^{1/6} \tag{1.39}$$

1.7 NORMAL FLOW

The condition of steady, uniform flow is called normal flow in the field of hydraulics. This simple flow provides a paradigm against which the gradually varied flow of the de St. Venant equations can be compared. For such a flow (1.35) reduces to

$$C_f\,U^2 = gHS \tag{1.40}$$

If the water discharge per unit width q_w, the bed slope S and the roughness height k_s are known, it is possible to solve for U and H from (1.34) and (1.40) with the aid of either (1.38) or (1.39). Boundary shear stress τ_b is then given by (1.36).

1.8 THE DE ST. VENANT EQUATIONS WITH LATERAL VARIATION

The above derivation considers only the case with streamwise and upward normal variation in the flow. The extension to transverse variation is straightforward. Where V denotes the depth-averaged transverse flow velocity and y denotes a boundary attached transverse coordinate, the resulting equations take the form

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(UH) + \frac{\partial}{\partial y}(VH) = 0 \tag{1.41}$$

$$\frac{\partial}{\partial t}(UH) + \frac{\partial}{\partial x}(U^2H) + \frac{\partial}{\partial y}(UVH) = -gH\frac{\partial H}{\partial x} - gH\frac{\partial \eta}{\partial x} - C_f\,(U^2 + V^2)^{1/2}\,\frac{U}{H} \tag{1.42}$$

$$\frac{\partial}{\partial t}(VH) + \frac{\partial}{\partial x}(UVH) + \frac{\partial}{\partial y}(V^2H) = -gH\frac{\partial H}{\partial y} - gH\frac{\partial \eta}{\partial y} - C_f\,(U^2 + V^2)^{1/2}\,\frac{V}{H} \tag{1.43}$$