CHAPTER 2
RELATIONS FOR SEDIMENT CONTINUITY

2.1 INTRODUCTION

Rivers move sediment in a variety of forms. At very steep slopes sediment may be moved in the form of debris flows. These flows consist of slurries for which the solid fraction by weight is of the same order of magnitude as the water fraction. Here, however, the sediment transport typical of rivers is considered.

In rivers the ratio of annual sediment discharge to annual water discharge is in general a very small number. That is, typically rivers move much more water than they do sediment. Sediment can be moved as either **wash load** or **bed material load**. Wash load consists of material that moves down the system without interacting with the channel bed. For example, in most sand bed streams the majority of the load is in the range of medium silt to clay, sizes that are present in the bed in only negligible quantities. In the case of gravel bed streams, sand may move through as a type of washload known as **throughput load**; the sand may deposit in the interstices of the gravel, but otherwise does not determine bed morphology.

These notes are concerned with bed material load. This is the portion of the load that is found on the bed of the stream and actively undergoes exchange between the bed and the water column. Bed material load may move as **bedload** or **suspended load**. Bedload consists of grains that slide, roll or hop (saltate) over the bed, with saltation being the most important mechanism. Turbulence plays an auxiliary role in the mechanics of bedload transport, which are largely governed by water drag and the role of bed collisions in converting streamwise particle momentum to upward particle momentum, so maintaining the saltation. Particles participating in bedload typically hug the bed as they move. The primary hydraulic factor determining the rate of bedload transport is the boundary shear stress τ_b. Bedforms such as dunes offer form drag that increase overall resistance without helping to transport sediment. With this in mind, it is that component of τ_b associated with skin friction, τ_{bs} that determines the bedload transport rate. Bedload tends to respond rather quickly to changes in boundary shear stress.

Particles participating in suspended load feel the turbulence, and can be wafted high into the water column by the action of the eddies. Only the rate of entrainment of such particles from the bed is determined by boundary shear stress. The transport rate itself is generally not locally in phase with variations in boundary shear stress.

Sand bed streams have median grain sizes D_{50} between 0.0625 and 2 mm, with the vast majority of cases falling in the range 0.1 to 0.8 mm. In such streams both bedload and suspended load play important roles in the bed material load, with suspended load dominating at flood conditions such as bankfull, when the river just spills out onto its floodplain. Gravel bed streams typically have a surface D_{50} between 10 and 200 mm. Such sizes are rarely suspended even at flood flows, so most of the sizes move as bedload. The sand in the interstices of the gravel can move as bedload or suspended load, with the latter playing the dominant role. Insofar as the sand is insufficient to cover the bed, however, it can be treated as throughput load.

The river bed changes over time in response to differential transport of sediment. In order to understand how this works, however, it is useful to specify some example sediment transport relations. The following notation is used below: q_b denotes the volume bedload transport per unit width, q_s denotes the volume suspended sediment transport rate per unit width, and $q_t = q_b + q_s$ denotes the total volume suspended sediment transport per unit width. The mass transport per unit width is obtained from any of these quantities by multiplying by the sediment density ρ_s.

2.2 EXAMPLE SEDIMENT TRANSPORT RELATIONS

A venerable relation for bedload transport is the relation of Meyer Peter and Muller (1948). It takes the form

$$q_b^* = 8(\tau^* - \tau_c^*)^{1.5} \tag{2.1}$$

where the dimensionless Einstein number is defined as

$$q_b^* = \frac{q_b}{\sqrt{RgD}\,D} \tag{2.2}$$

and the dimensionless Shields stress is defined as

$$\tau^* = \frac{\tau_b}{\rho R g D} \tag{2.3}$$

In the above equations D denotes a characteristic grain size of the bed material (e.g. D_{50}), R is given by

$$R = \frac{\rho_s}{\rho} - 1 \tag{2.4}$$

and τ_c^* denotes a critical Shields stress for the onset of sediment motion. For most natural sediments *R* is close to 1.65; in the Meyer Peter and Muller relation the experimentally determined value of τ_c^* is 0.047. The equation was originally developed for gravel bed streams with negligible bedforms. If bedforms are present, τ_b must be replaced with only the portion due to skin friction τ_{bs} in the definition of (2.3).

The Meyer Peter and Muller relation has been superseded by more accurate relations. This notwithstanding, it serves as a prototype for bedload transport relations. The impelling force is the drag force on the bed, quantified in the relation by the numerator of the Shields stress. The resisting force is the submerged weight of the particle, quantified in terms of the denominator of the Shields stress.

In the case of suspended sediment, the quantity that can be expected to vary with boundary shear stress is the volume rate of entrainment of sediment into suspension per unit time per unit bed area *E*. This parameter can be replaced by a dimensionless entrainment rate E by the intermediary of the terminal fall velocity in still water v_s of the sediment in question;

$$E = v_s \mathrm{E} \qquad (2.5)$$

A sample relaton for E versus Shields stress is that of Smith and McLean: it takes the form

$$\mathrm{E} = 0.65 \frac{\gamma_o \left(\frac{\tau^*}{\tau_c^*} - 1\right)}{1 + \gamma_o \left(\frac{\tau^*}{\tau_c^*} - 1\right)} \qquad \gamma_o = 0.0024 \qquad (2.6)$$

Since bedforms are particularly prevalent in the case of sand bed streams, the Shields stress in the above relation must refer only to skin friction.

2.3 EQUATION OF SEDIMENT CONTINUITY

The equation of sediment continuity was first delineated by the Austrian researcher Exner near the beginning of the 20^{th} century, and is named in honor of him. First a one-dimensional case is considered. As with the de St. Venant equations, x represents a boundary attached downstream coordinate. Sediment mass balance is expressed in volume form (by dividing the mass balance by ρ_s) below;

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_b}{\partial x} + v_s(\bar{c}_b - E) \qquad (2.7)$$

In the above relation λ_p denotes the porosity of the bed deposit and $\bar{c}_b$ denotes the volume concentration of suspended sediment, averaged over turbulence, just above the bed. Bedload is seen to act as a flux term and suspended sediment as a source/sink term. The bed elevation of a control volume increases if more bedload is entering than exiting, resulting in net deposition of bedload. Noting that $v_s\ \bar{c}_b$ denotes the volume flux of suspended sediment settling on to the bed and v_s E denotes the volume flux of entrainment of bed sediment into suspension, bed elevation increases due to the net deposition of suspended sediment if $\bar{c}_b > E$.

When the length scales of interest are large compared to the relaxation distance (U/v_s) H associated with the settling of suspended sediment, (2.7) can be rigorously reduced to a simpler form,

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_t}{\partial x} \qquad q_t = q_b + q_s \qquad (2.8)$$

Such a formulation is usually not adequate to explain such features as ripples and dunes which have a rather short wavelength. It is partially adequate to explain bars, and is generally adequate to explain bed aggradation and degradation (increases and decreases in bed elevation) over many kilometers in response to e.g. a channel cutoff or diversion. To implement the equation q_s is determined as a function of τ_b and other parameters for equilibrium conditions, and this relation is applied to the mild disequilibrium characteristic of bed changes which are distributed over long distances.

In order to explain two-dimensional features such as bars, however, it is necessary to consider a two-dimensional form for the Exner equation. The two-dimensional form for (2.7) is

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_{bx}}{\partial x} - \frac{\partial q_{by}}{\partial y} + v_s(\bar{c}_b - E) \qquad (2.9)$$

where (q_{bx}, q_{by}) denotes the bedload transport vector tangential to the boundary. The bedload transport function is usually generalized to the two-dimensional case such that the vector of bedload transport is parallel to the vector of boundary shear stress (τ_{bx}, τ_{by}). That is, if

$$q_b = f(\tau_b) \qquad (2.10)$$

denotes the scalar relation for the one-dimensional case, the two-dimensional generalization is

$$q_{bx} = f(|\tau_b|)\frac{\tau_{bx}}{|\tau_b|} \qquad q_{by} = f(|\tau_b|)\frac{\tau_{by}}{|\tau_b|} \qquad |\tau_b|^2 = \tau_{bx}^2 + \tau_{by}^2 \tag{2.11}$$

The above equations must, however, be adjusted to account for the tendency for bedload to move down streamwise or transverse bed slopes when they are substantial. Such is the case in a treatment of river meandering. The appropriate forms of τ_{bx} and τ_{by} consistent with (1.42) and (1.43) are

$$\tau_{bx} = \rho C_f (U^2 + V^2)^{1/2} U \qquad \tau_{by} = \rho C_f (U^2 + V^2)^{1/2} V \tag{2.12}$$

These forms must also be modified in a treatment of meandering in order to account for the effect of secondary flow associated with bends.

In the two dimensional case the dimensionless entrainment rate E must be taken as a function of the magnitude of the shear stress vector $|\tau_b|$ rather than any component of that vector.

The generalization of (2.8) to the two dimensional case is straightforward;

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_{tx}}{\partial x} - \frac{\partial q_{ty}}{\partial y} \tag{2.13}$$

where q_{tx} and q_{ty} denote the x and y components of the total bed material load (bedload + suspended load).

Various other forms of sediment continuity could be introduced, e.g. a formulation for mixtures of sediment sizes. One form of relevance to these notes is the Exner equation in the case of tectonic subsidence. It is this subsidence that creates the accommodation space for such large features as alluvial fans. For such large scales a formulation using total load is adequate. Fans are two-dimensional features, but approximate radial symmetry allows for an effective one-dimensional treatment. The fan is built up by constantly aggrading and avulsing (jumping) channels. Fan width B_f typically increases with radial distance r down the fan. Let B_c denote the channel width at a point and q_t denote the effective mean bed material transport rate in the radial direction, i.e. shorthand for q_{tr}. The equation of sediment continuity for this problem takes the form.

$$(1-\lambda_p)B_f\left(\frac{\partial \eta}{\partial t} + \sigma\right) = -\frac{\partial B_c q_t}{\partial r} \tag{2.14}$$

where σ denotes the velocity of subsidence of the crust of the earth below the fan.

2.4 CONTINUITY OF SUSPENDED SEDIMENT

In order to apply the above formulation to cases with suspended sediment, it is necessary to know the near-bed mean concentration of suspended sediment $\bar{c}_b$. This requires an accounting of sediment balance everywhere in the water column.

Let $c(x_i, t)$ denote the instantaneous volume concentration of suspended sediment in the water column (volume of sediment/volume of sediment+water) as a function of position x_i and time t, and let u_{si} denote the local velocity of the sediment particles. Local mass balance reduces to the following volume form;

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x_i}(u_{si} c) = 0 \qquad (2.15)$$

For sufficiently fine sediment the sediment velocity can be accurately approximated as the sum of the fluid velocity and the terminal fall velocity in still water;

$$u_{si} = u_i - n_{vi} v_s \qquad (2.16)$$

where again n_{vi} denotes an upward vertical coordinate. Substituting (2.16) into (2.15) and averaging over turbulence, it is found that

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial}{\partial x}(\bar{u}\,\bar{c}) + \frac{\partial}{\partial y}(\bar{v}\,\bar{c}) + \frac{\partial}{\partial z}(\bar{w}\,\bar{c}) - v_s \frac{\partial \bar{c}}{\partial z} = -\frac{\partial}{\partial x}\overline{u'c'} - \frac{\partial}{\partial y}\overline{v'c'} - \frac{\partial}{\partial z}\overline{w'c'} \qquad (2.17)$$

where the overbar again denotes averaging and the primed quantities again denote fluctuations. The Reynolds fluxes of suspended sediment are the quantities $\overline{u'c'}$ etc. on the right-hand side of the equation. These fluxes are typically expressed in terms of a formulation using an eddy viscosity. In the above relation the upward normal coordinate is taken to deviate only slightly from the vertical, an assumption consistent with the assumption of small bed slope S.

The net rate of sediment accumulation on the bed is determined by the sediment flux crossing the bed. Approximating the bed as facing vertically upward, this flux is given by

$$F_{sz}\big|_{bed} = (-v_s \bar{c} + \overline{w'c'})\big|_{bed} \qquad (2.18)$$

Recalling that $v_s \; \overline{c}_b$ denotes the rate of depositional flux on the bed, it follows that the volume rate of entrainment of sediment from the bed into suspension per unit bed area per unit time *E* is given by

$$E = v_s E = \overline{w'c'}\Big|_{bed} \tag{2.19}$$

Here the vertical Reynolds flux is closed as

$$\overline{w'c'} = -\nu_e \frac{\partial \overline{c}}{\partial z} \tag{2.20}$$

where ν_e denotes a kinematic eddy diffusivity. The bottom boundary condition for the the sediment suspension field thus takes gradient form;

$$-\nu_e \frac{\partial \overline{c}}{\partial z}\bigg|_{bed} = v_s E \tag{2.21}$$

where E is a specified function of shear stress, as outlined above.

Attention is now focused on steady, uniform equilibrium suspensions. For this case (2.17) reduces to

$$\frac{d}{dz} F_{sz} = 0 \qquad F_{sz} = -v_s \overline{c} + \overline{w'c'} \tag{2.22}$$

Integrating under the condition of vanishing sediment flux at the water surface, it is found that

$$F_{sz} = 0 \tag{2.23}$$

The combination of (2.19), (2.22) and (2.23) specifies the following ordinary differential equation,

$$\nu_e \frac{d\overline{c}}{dz} + v_s \overline{c} = 0 \tag{2.24}$$

subject to the boundary condition (2.21). In the case of an equilibrium suspension, however, it is seen from (2.21), (2.22) and (2.23) applied to the bed that

$$\bar{c}_b = E \tag{2.25}$$

Rouse used the Prandtl analogy to obtain ν_e from a momentum formulation. While in retrospect there are several flaws in his formulation, it grasps the essence of the problem. Specifically, he combined the linear shear stress distribution associated with steady, uniform flow

$$\tau_{Rxz} = \tau_b (1 - \frac{z}{H}) \tag{2.26}$$

with an eddy diffusivity formulation for momentum

$$\tau_{Rxz} = -\rho \overline{u'w'} = \rho \nu_e \frac{\partial \bar{u}}{\partial z} \tag{2.27}$$

and the logarithmic flow velocity of (1.38) to obtain the evaluation

$$\nu_e = \kappa u_* H (1 - \frac{z}{H}) \tag{2.28}$$

where $\kappa = 0.4$ denotes the Karman constant. Integrating (2.24) with the aid of (2.21), (2.25) and (2.28) it is found that

$$\bar{c} = \bar{c}_b [\frac{(1-\varsigma)/\varsigma}{(1-\varsigma_b)/\varsigma_b}]^{Z_r} \tag{2.29}$$

where $\zeta = z/H$, $\zeta_b = b/H$ where $b/H \ll 1$ and Z_r is the Rouse number, given by

$$Z_r = \frac{v_s}{\kappa u_*} \tag{2.30}$$

and $\bar{c}_b$ is evaluated from (2.25) and an appropriate entrainment relation.

Note that $\bar{c}_b$ is not evaluated precisely at the bed, but rather a small distance $z = b$ above this. The reason for this is that the form of (2.28) predicts vanishing eddy viscosity precisely at the bed. Setting the boundary condition slightly above the bed circumvents a complex but ultimately unimportant boundary layer problem associated with near-bed turbulence in a hydraulically rough turbulent field.

Equation (2.29) predicts that the sediment concentration should be concentrated very close to the bed for large Rouse numbers, i.e. coarse sediment, and should be essentially uniform in the vertical for small Rouse numbers, i.e. fine sediment. This is in agreement with the following empirical criterion for the onset of significant sediment suspension;

$$\frac{u_*}{v_s} = 1 \tag{2.31}$$

That is, the shear velocity must exceed the fall velocity for significant sediment suspension.

The Rousean theory has been modified over the years, but captures the essence of equilibrium suspended sediment mechanics. The volume suspended sediment load per unit width q_s is in general given by

$$q_s = \int_0^H \overline{uc}\,dz \tag{2.32}$$

This can be evaluated from the equilibrium case by substituting in (2.29) and (1.38) and integrating.