CHAPTER 6
AN APPLICATION

6.1 INTRODUCTION

The application explored here is that of a recirculating flume. The complicated approach to equilibrium suggests similar behavior in field rivers under certain circumstances.

6.2 SEDIMENT FEED AND RECIRCULATING FLUMES

Flumes provide model rivers that allow for the testing of general hypotheses. Flumes come in two flavors; sediment-feed and recirculating. In a sediment-feed flume, the water discharge and the sediment feed rate are set at the upstream end of the flume; the flow develops toward an equilibrium slope, velocity and depth. In a recirculating flume, the total amount of water and sediment in the flume are held constant, and the water and sediment returned to the upstream end via a pump that maintains constant water discharge. The flow evolves toward equilibrium sediment discharge, slope and velocity.

Since flumes typically have constant widths, the problem can be formulated in terms of quantities per unit width. In the sediment-feed flume, water discharge per unit width q_w and total volume sediment feed rate q_t are set; U, H and S develop toward equilibrium. In a recirculating flume, q_w and average flow depth H are set; q_t, U and S develop toward equilibrium.

That such an equilibrium exists can be seen by applying a sediment transport formulation to normal flow. The sediment transport relation is taken to be of the generic form

$$q_t^* = f(\tau^*) \qquad (6.1)$$

where f is an appropriately specified function, e.g. the Meyer-Peter Muller relation of (2.1). Note that

$$\tau^* = \frac{\tau_b}{\rho R g D} \qquad \tau_b = \rho C_f U^2 \qquad (6.2a,b)$$

For simplicity the friction relation is assumed to be (1.38), or

$$C_f^{-1/2} = 2.5\,\ell n(11\frac{H}{2.5\,D}) \qquad (6.3)$$

The relations for water mass and momentum conservation for normal flow are (1.34) and (1.40);

$$UH = q_w \qquad C_f U^2 = gHS \qquad (6.4a,b)$$

In a sediment-feed flume q_t and q_w and grain size D are specified; (6.1) to (6.4) specify 6 equations in the 6 unknowns τ^*, τ_b, C_f, U, H and S, allowing for a solution. In a recirculating flume q_w and H are specified; the same 6 equations allow the specification of τ^*, τ_b, C_f, U, q_t and S.

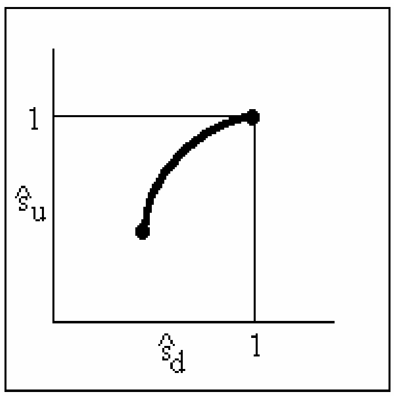

Figure 6.1 Approach to equilibrium for a sediment-feed flume.

The way equilibrium is reached is, however, dramatically different between the two types of flumes. Let S_e denote the equilibrium slope, and let

$$\hat{s} = \frac{S(x,t)}{S_e} \qquad (6.5)$$

denote a normalized slope such that $\hat{s} = 1$ at equilibrium conditions. Suppose that the initial value of slope in a sediment-feed flume is set to be a spatially constant value that is too low relative to the sediment feed rate and water discharge, i.e. $\hat{s} = \hat{s}_i < 1$. Slope should first increase at the upstream end in response to aggradation driven by proximal sediment deposition. In time, the effect should propagate downstream until $\hat{s} = 1$ everywhere. A phase diagram of the response can be plotted in terms of slope near the upstream end of the flume $\hat{s}_u$ versus slope near the downstream end of the flume $\hat{s}_d$.

The simple result of Figure 6.1 is obtained for a sediment-feed flume. It is seen that the approach to the steeper equilibrium is uniform and straightforward.

6.3 FORMULATION FOR RECIRCULATING FLUMES

In the case of a recirculating flume the approach is quite different, and much more interesting. The condition that the sediment discharge at the upstream end of the flume must always equal the value at the downstream end, regardless of hydraulic conditions at the upstream end, introduces a critical mismatch that makes the approach to equilibrium quite complicated.

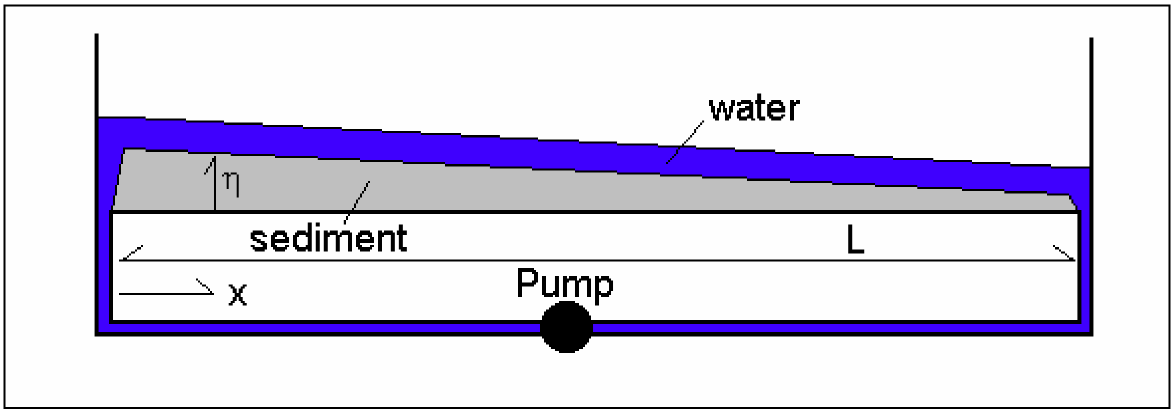

Figure 6.2 Illustration of a recirculating flume.

A recirculating flume is illustrated above. The length of the flume is L; the x coordinate points downstream from the upstream end of the flume. The elevation of sediment above the bed is $\eta(x, t)$ where t denotes time, and flow depth is $H(x, t)$. Here the case of subcritical flow in the flume is considered. The flume has constant width B, and the water discharge per unit width q_w is assumed to be constant.

Assuming that the flume does not leak, the total amount of water in the flume is conserved, as well as the total amount of sediment. Assuming that a) end effects are negligible, b) the total amount of water stored in the recirculating pipe is constant and c) the total amount of sediment conserved in the recirculating pipe is negligible, it follows that

$$\int_0^L H dx = C_1 \qquad \int_0^L \eta dx = C_2 \qquad (6.6a,b)$$

where C_1 and C_2 are constants.

Momentum balance of the open-channel flow is obtained by reducing (1.35) with (1.34) and (1.8) to obtain

$$\frac{\partial H}{\partial x} = \frac{-\frac{\partial \eta}{\partial x} - C_f \frac{q_w^2}{gH^3}}{1 - \frac{q_w^2}{gH^3}} \tag{6.7}$$

where the friction coefficient C_f is here assumed to be constant for simplicity. If a downstream depth H_{end} is assumed, the above equation can be integrated upstream to find the complete depth profile for any given values of q_w and the bed profile η.

Sediment is assumed to be transported solely as bedload in accordance to the Meyer-Peter Muller relation;

$$q_b^* = \begin{cases} 0 \quad \text{if} \quad \tau^* < 0.047 \\ 8(\tau^* - 0.047)^{1.5} \quad \text{if} \quad \tau^* > 0.047 \end{cases} \tag{6.8}$$

Here q_b^* and τ^* have their usual meanings, i.e. $q_b^* = q/[(RgD)^{1/2}D]$ and $\tau^* = \tau_b/(\rho RgD)$ and bed shear stress τ_b is given by the relation

$$\tau_b = \rho C_f u^2 = \rho C_f \frac{q_w^2}{H^2} \tag{6.9}$$

Local sediment mass balance is satisfied via the Exner equation of sediment continuity; where λ_p denotes the porosity of the bed sediment,

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_b}{\partial x} \tag{6.10}$$

Integrating this equation from x = 0 to x = L gives the result

$$(1-\lambda_p)\frac{d}{dt}\int_0^L \eta dx = q_b(0) - q_b(L) \tag{6.11}$$

Between (6.11) and (6.6b) it is seen that

$$q_b(0) = q_b(L) \tag{6.12}$$

i.e. the boundary condition on sediment transport must be cyclic.

6.4 NORMAL FLOW IN A RECIRCULATING FLUME

The flow and sediment transport in the flume has a steady, uniform solution corresponding to normal flow. Let S_e be the constant bed (and water surface) slope associated with this equilibrium normal flow, and let H_e be the normal depth. Under these conditions (1a,b) reduces to

$$H_e L = C_1 \qquad \eta_{ee} L + \frac{1}{2} S_e L^2 = C_2 \tag{6.13a,b}$$

where η_{ee} denotes the elevation of the bed at the downstream end of the flume. In addition, (6.7) reduces to

$$S_e = C_f \frac{q_w^2}{g H_e^3} \tag{6.14}$$

Where q_{be} denotes the bedload transport rate at normal flow, the sediment transport rate can be abbreviated to

$$q_{be} = f(\tau_{be}) \qquad \tau_{be} = \rho C_f \frac{q_w^2}{H_e^3} \tag{6.15}$$

where the functional form $f(\tau_b)$ is specified by (6.8) (6.16)

Assuming that q_w and C_f are given, (6.13a,b), (6.14), (6.15) and (6.16) specify five equations in five unknowns: H_e, S_e, η_{ee}, τ_{be} and q_{be}. It is thus always possible to solve for the normal flow.

6.5 DIMENSIONLESS FORMULATION

The equations may be placed in dimensionless form as follows;

$$H = H_e \hat{h} \qquad q_b = q_{be} \hat{q} \qquad x = L\hat{x} \qquad t = (1-\lambda_p)\frac{S_e L^2}{q_{be}} \hat{t} \qquad \eta = \overline{\eta} + S_e L \hat{\eta}$$

where $\overline{\eta}$ denotes the mean bed elevation, which is constrained to be constant in accordance with (6.6b). With the above relations, (6.6a, b) reduce to

$$\int_0^1 \hat{h} d\hat{x} = 1 \qquad \int_0^1 \hat{\eta} d\hat{x} = 0 \tag{6.17a,b}$$

In addition, (6.7) reduces to

$$\alpha_1 \frac{\partial \hat{h}}{\partial \hat{x}} = \frac{-\dfrac{\partial \hat{\eta}}{\partial \hat{x}} - \hat{h}^{-3}}{1 - \mathbf{Fr_e^2}\, \hat{h}^{-3}} \tag{6.18}$$

where

$$\mathbf{Fr}_e^2 = \frac{q_w^2}{g H_e^3} \tag{6.19}$$

denotes the Froude number at normal flow (here assumed to be less than unity) and

$$\alpha_1 = \frac{1}{S_e}\frac{H_e}{L} \tag{6.20}$$

The relation for sediment transport can be reduced to

$$\hat{q} = \begin{cases} 0 \quad \text{if} \quad \hat{h}^{-2} < 0.047/\tau_n^* \\ \left(\dfrac{\hat{h}^{-2} - 0.047/\tau_e^*}{1 - 0.047/\tau_e^*}\right)^{3/2} \quad \text{if} \quad \hat{h}^{-2} > 0.047/\tau_n^* \end{cases} \tag{6.21}$$

where

$$\tau_e^* = \frac{\tau_{be}}{\rho R g D} \tag{6.22}$$

denotes the Shields stress at normal flow. Finally, the Exner equation takes the dimensionless form

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} = -\frac{\partial \hat{q}}{\partial \hat{x}} \tag{6.23}$$

subject to the boundary condition

$$\hat{q}(1) = \hat{q}(0) \tag{6.24}$$

It is seen from the nondimensionalization that bed slope S is given by

$$S = -\frac{\partial \eta}{\partial x} = -S_e \frac{\partial \hat{\eta}}{\partial \hat{x}}$$

Thus at normal flow

$$\frac{\partial \hat{\eta}_e}{\partial \hat{x}} = -1$$

This combined with (6.17b) gives

$$\hat{\eta}_e = \frac{1}{2} - \hat{x} \tag{6.25}$$

Initially we could assume any bed satisfying (6.17b), but here for simplicity we assume

$$-\left.\frac{\partial \hat{\eta}}{\partial \hat{x}}\right|_{\hat{t}=0} \equiv \hat{s}_i \qquad \left.\hat{\eta}\right|_{\hat{t}=0} = \hat{s}_i\left(\frac{1}{2} - \hat{x}\right) \tag{6.26}$$

Thus if $\hat{s}_i$ = 0.5 then the initial bed slope is spatially constant and half the value of the normal bed slope.

The method of numerical solution of for the evolution toward equilibrium is now outlined. The dimensionless parameters α_1. $\mathbf{Fr}_e$ and τ_e^* must be known in order to implement a solution. Suppose that at some time we know the bed elevation profile $\hat{\eta}(\hat{x})$. If one knew the downstream value of $\hat{h}$, i.e. $\hat{h}_{end}$ at $\hat{x} = 1$, one could integrate (6.18) upstream to determine the depth everywhere. We would have to iterate for the value of $\hat{h}_{end}$ which gave a depth profile satisfying (6.17a). This is most conveniently done using a shooting method.

6.6 SHOOTING SCHEME FOR WATER FLOW

Having found $\hat{h}(\hat{x})$ at the time in question, one could evaluate the sediment transport rate $\hat{q}$ from (6.21) everywhere except the farthest upstream node at $\hat{x} = 0$, where one would impose (6.24). The new bed elevation at some time $\hat{t} + \Delta\hat{t}$ later is found from (6.23).

The shooting scheme can be outlined as follows. Given the bed profile, for each downstream value $\hat{h}_{end}$ at $\hat{x} = 1$ it is possible to integrate (6.18) upstream and find a solution $\hat{h} = \hat{h}(\hat{x}, \hat{h}_{end})$. Only one value of $\hat{h}_{end}$ will satisfy (6.17a). To find it one defines

$$\hat{H} = \frac{\partial \hat{h}}{\partial \hat{h}_{end}}$$

The two equations to be solved then become (according to the shooting method)

$$\alpha_1 \frac{\partial \hat{h}}{\partial \hat{x}} = \frac{-\frac{\partial \hat{\eta}}{\partial \hat{x}} - \hat{h}^{-3}}{1 - \mathbf{Fr_e^2}\,\hat{h}^{-3}} \qquad \hat{h}(1) = \hat{h}_{end}$$

$$\alpha_1 \frac{\partial \hat{H}}{\partial \hat{x}} = \frac{3\hat{h}^{-4}}{1 - \mathbf{Fr_e^2}\,\hat{h}^{-3}}\left[1 + \mathbf{Fr_e^2}\,\frac{\frac{\partial \hat{\eta}}{\partial \hat{x}} + \hat{h}^{-3}}{1 - \mathbf{Fr_e^2}\,\hat{h}^{-3}}\right]\hat{H} \qquad \hat{H}(1) = 1 \tag{6.27}$$

The condition (12a) becomes

$$\varphi(\hat{h}_{end}) = \int_0^1 \hat{h}(\hat{x}, \hat{h}_{end})d\hat{x} - 1 = 0$$

or invoking Newton-Raphson,

$$\hat{h}_{end}^{new} = \hat{h}_e - \frac{\varphi(\hat{h}_{end})}{\varphi'(\hat{h}_{end})} \qquad \varphi'(\hat{h}_{end}) = \frac{\partial}{\partial \hat{h}_{end}} \int_0^1 \hat{h}(\hat{x}, \hat{h}_{end})d\hat{x} = \int_0^1 \hat{H} d\hat{x}$$

or finally

$$\hat{h}_{end}^{new} = \hat{h}_{end} - \frac{\int_0^1 \hat{h} d\hat{x} - 1}{\int_0^1 \hat{H} d\hat{x}} \tag{6.28}$$

6.7 RESULTS OF NUMERICAL IMPLEMENTATION

A numerical implementation of the above will be illustrated during class. The approach to equilibrium will be demonstrated to be complicated and rich in behavior. The problem illustrates the rather complicated response that can occur in rivers when disjointed conditions are imposed, e.g. at tributary confluences.