EQUATIONS FOR CONSERVATION OF BED SEDIMENT (Exner Equations)

SOME PARAMETERS

q_b = volume bedload transport rate per unit width [L^2T^{-1}]

q_s = volume suspended load transport rate per unit width [L^2T^{-1}]

$q_t = q_b + q_s$ = volume bed material transport rate per unit width [L^2T^{-1}]

$g_b = \rho_s q_b$ = mass bedload transport rate per unit width [$ML^{-1}T^{-1}$]

(similar definitions for g_s, g_t)

η = bed elevation [L]

λ_p = porosity of sediment in bed deposit [1]

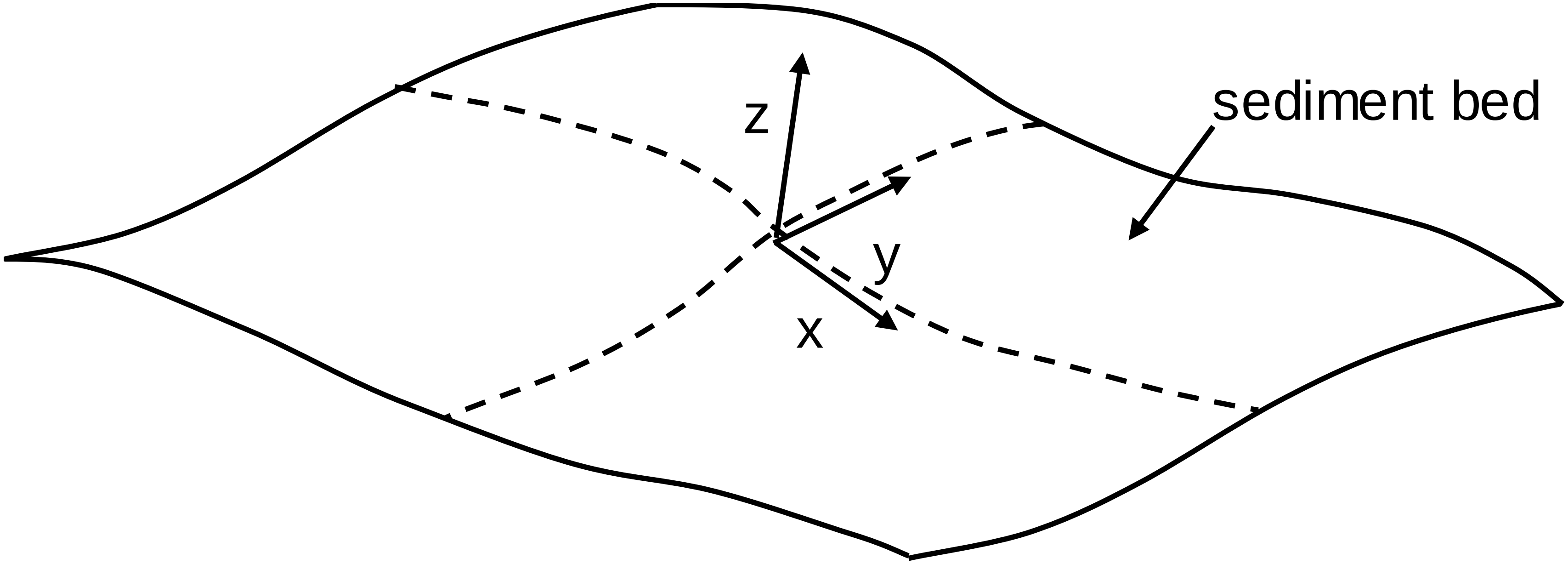
(volume fraction of bed sample that is holes rather than sediment: 0.25 ~ 0.55 for noncohesive material)

g = acceleration of gravity [L/T^2]

x = boundary-attached streamwise coordinate [L]

t = time [T]

COORDINATE SYSTEM



x = nearly horizontal boundary-attached “streamwise” coordinate [L]

y = nearly horizontal boundary-attached “transverse” coordinate [L]

z = nearly vertical coordinate upward normal from boundary [L]

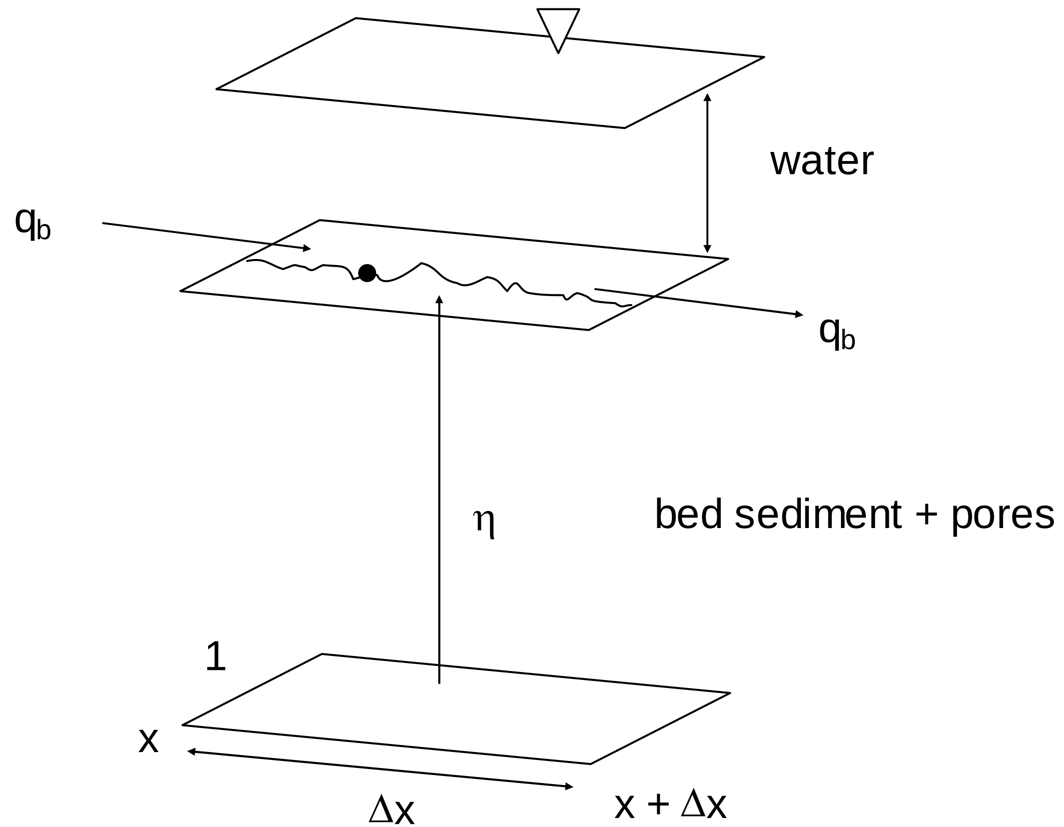
CASE OF 1D BEDLOAD ONLY

$$\frac{\partial}{\partial t}\left[\rho_s(1-\lambda_p)\eta\right]\Delta x \cdot 1 = \rho_s\left[q_b\big|_x - q_b\big|_{x+\Delta x}\right]\cdot 1$$

or thus

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_b}{\partial x}$$

This is the original form derived by Exner



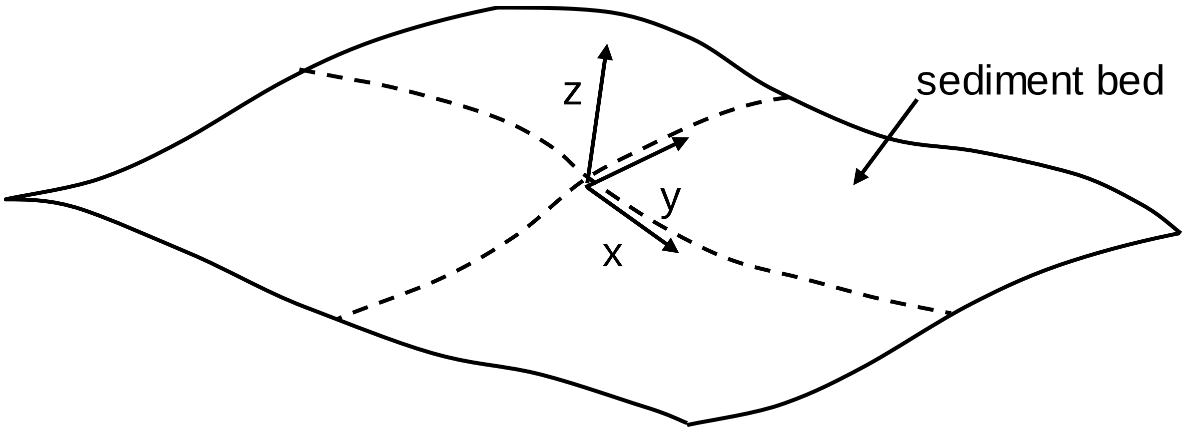
2D GENERALIZATION, BEDLOAD ONLY

$$\vec{q}_b = q_{bx}\hat{e}_x + q_{by}\hat{e}_y$$

where

$$\hat{e}_x, \hat{e}_y$$

denote unit vectors in the x and y directions



$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\vec{\nabla} \cdot \vec{q}_b$$

CASE OF 1D BEDLOAD + SUSPENDED LOAD

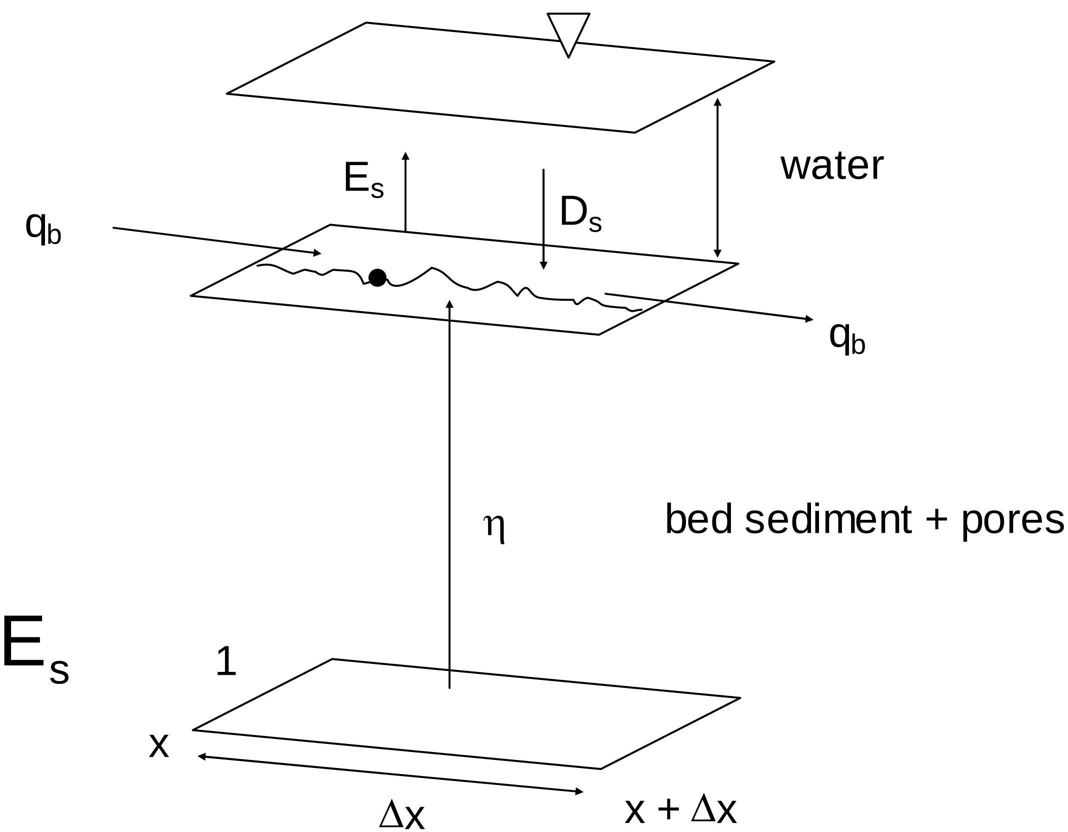
E_s = volume rate per unit time per unit bed area that sediment is entrained from the bed into suspension [LT^{-1}]

D_s = volume rate per unit time per unit bed area that sediment is deposited from the water column onto the bed [LT^{-1}]

$$\frac{\partial}{\partial t}\left[\rho_s(1-\lambda_p)\eta\right]\Delta x\cdot 1 = \rho_s\left[q_b\big|_x - q_b\big|_{x+\Delta x}\right]\cdot 1 + (D_s - E_s)\Delta x\cdot 1$$

or thus

$$(1-\lambda_p)\frac{\partial\eta}{\partial t} = -\frac{\partial q_b}{\partial x} + D_s - E_s$$

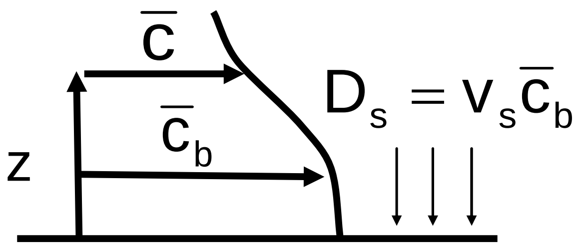


EVALUATION OF D_s AND E_s

Let $\bar{c}$(x, y, z, t) denote the volume concentration of sediment c in suspension at (x, y, z, t), averaged over turbulence. Here c = (sediment volume)/(water volume + sediment volume).

In the case of a dilute suspension of non-cohesive material,

$$D_s = v_s \bar{c}_b$$



where $\bar{c}_b$ denotes the near-bed value of c.

Similarly, a dimensionless entrainment rate E can be defined such that

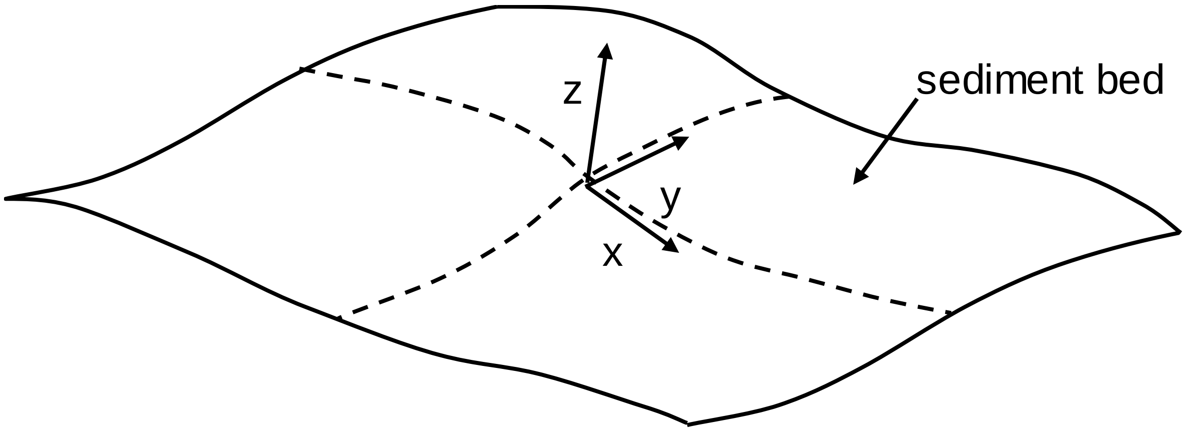
$$E_s = v_s E$$

Thus

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_b}{\partial x} + v_s\left(\bar{c}_b - E\right)$$

2D GENERALIZATION, BEDLOAD + SUSPENDED LOAD

$$\vec{q}_b = q_{bx}\hat{e}_x + q_{by}\hat{e}_y$$



$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\vec{\nabla}\cdot\vec{q}_b + v_s(c_b - E)$$

FURTHER PROGRESS

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\vec{\nabla}\cdot\vec{q}_b + v_s(c_b - E)$$

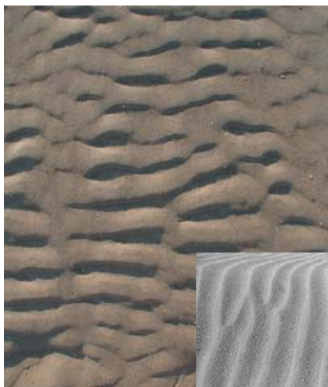
In order to make further progress, we must

- Develop a means for computing the bedload transport rate as a function of the flow;
- Develop a means for computing the entrainment rate into suspension as a function of the flow;
- Develop a model for tracking the concentration c of sediment in suspension.

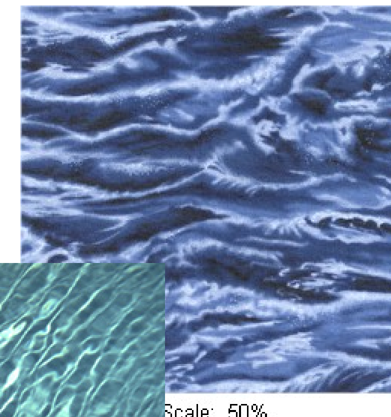
The key parameter turns out to be **boundary shear stress** $\vec{\tau}_b$ **.**

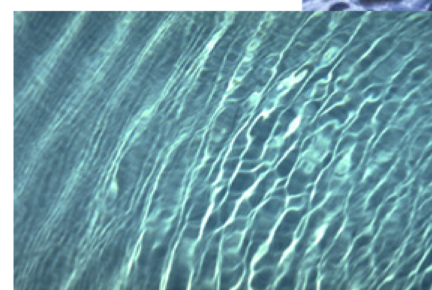
AND TO COMPUTE $\vec{\tau}_b$, WE HAVE TO KNOW SOMETHING ABOUT FLUID MECHANICS!!!

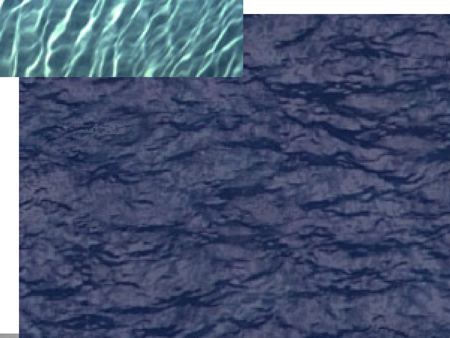
Morphodynamics

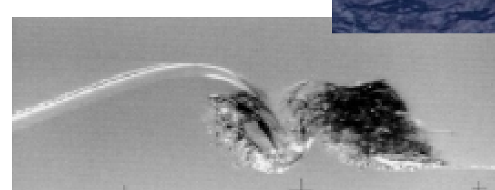


Fluid dynamics









OVERVIEW OF FLUID MECHANICS

Consider the case of turbulent water above a granular bed carrying a dilute suspension of sediment (the normal case in morphodynamics).

Again coordinates x and y are boundary-attached parallel to to the sediment bed, and z is upward normal (nearly vertical) from the bed. In addition, $\vec{u}$ denotes the instantaneous vector of water velocity, p denotes the instantaneous pressure and $\hat{k}$ denotes a unit upward vertical vector.

$$\frac{\partial u_i}{\partial x_i} = 0$$

continuity

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - g k_i$$

Navier-Stokes

Approximating instantaneous sediment velocity $\vec{u}_s$ as $\vec{u}_s = \vec{u} - v_s \hat{k}$

$$\frac{\partial c}{\partial t} + \frac{\partial\left[\left(u_i - v_s k_i\right)c\right]}{\partial x_i} = 0$$

Conservation of sediment in suspension

DECOMPOSE INTO MEAN AND FLUCTUATING PARTS AND AVERAGE OVER TURBULENCE

$$u_i = \overline{u}_i + u' , \quad p = \overline{p} + p' , \quad c = \overline{c} + c'$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$ continuity

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \nabla^2 \overline{u}_i - g k_i + \frac{1}{\rho}\frac{\partial \tau_{Rij}}{\partial x_j}$$ Reynolds-averaged Navier-Stokes

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x_i}\left[\left(\overline{u}_i - v_s k_i\right)\overline{c}\right] = -\frac{\partial F_{Ri}}{\partial x_i}$$ Reynolds-averaged conservation of sediment in suspension

Reynolds stress tensor and Reynolds suspended sediment flux vector

$$\tau_{Rij} = -\rho \overline{u_i' u_j'} , \quad F_{Ri} = \overline{c' u_i'}$$

The above equations can be closed with appropriate assumptions for τ_{Rij} and F_{Ri}: e.g. k-ε, k-ω, Mellor-Yamada (1974) etc.

EVALUATION OF BOUNDARY SHEAR STRESS AND ENTRAINMENT RATE OF SEDIMENT FROM THE BED

Let i = 1, 2 denote the x, y (boundary-parellel) directions, i = 3 denote the upward normal (nearly vertical) direction. The bed shear stress $\vec{\tau}_b$ and entrainment rate of bed sediment into suspension E_s are given as

$$\vec{\tau}_b = \tau_{bx}\hat{e}_x + \tau_{by}\hat{e}_y$$

$$\tau_{bx} = \tau_{R13}\Big|_{z=b} = -\rho\overline{u'w'}\Big|_{z=b}$$

$$\tau_{by} = \tau_{R23}\Big|_{z=b} - \rho\overline{v'w'}\Big|_{z=b}$$

$$E_s = v_s E = F_{R3}\Big|_{z=b} = \overline{c'w'}\Big|_{z=b}$$

where z = b is a near-bed location such that z/H << 1

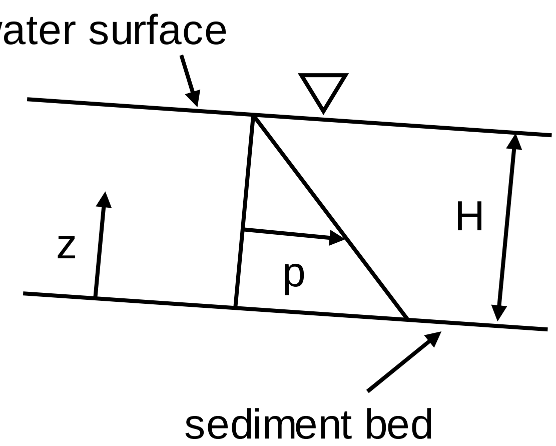
SLENDER FLOW APPROXIMATION

•Assume slender flow: flow changes much more strongly in upward normal direction than in any direction parallel to the flow. This allows pressure distribution to be approximated as hydrostatic, and also allows for neglect of most terms in Reynolds stress and flux.

•Neglect viscous terms for a fully turbulent flow. (Later may to have to reinsert some of them in a thin layer near the wall.)

•Exploit the approximation that k_i is nearly upward vertical

$$\overline{p} \cong \rho g(H - z)\,, \quad k_i \cong \left(-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1 \right)$$



where H denotes flow depth.
Note: we've also neglected wind shear stress and the Coriolis acceleration.

SLENDER FLOW APPROXIMATION contd.

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\frac{\partial H}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} - g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + \overline{w}\frac{\partial \overline{v}}{\partial z} = -g\frac{\partial H}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial y} - g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} + \overline{v}\frac{\partial \overline{c}}{\partial y} + \left(\overline{w} - v_s\right)\frac{\partial \overline{c}}{\partial z} = -\frac{\partial F}{\partial z}$$

$$\tau_{xz} = -\rho\overline{u'w'}\,, \quad \tau_{yz} = -\rho\overline{v'w'}\,, \quad F = \overline{c'w'}$$

ST. VENANT SHALLOW WATER EQUATIONS

- Define depth-averaged velocities U and V and concentration C such that:

$$UH = \int_0^H \overline{u}\,dz\,,\quad VH = \int_0^H \overline{v}\,dz\,,\quad CH = \int_0^H \overline{c}\,dz$$

- Integrate over the depth of flow using the following boundary conditions, where $\hat{n}_s$ denotes a unit vector normal to the water surface:

$$\overline{u}\big|_{z=0} = 0\,,\quad \overline{v}\big|_{z=0} = 0\,,\quad \frac{\partial H}{\partial t} + \overline{u}\big|_{z=H}\frac{\partial H}{\partial x} + \overline{v}\big|_{z=H}\frac{\partial H}{\partial y} - \overline{w}\big|_{z=H} = 0$$

Flow velocity vanishes at bed — *Water surface is streamline*

$$\overline{u}\,\overline{c}\big|_{z=H} n_{sx} + \overline{v}\,\overline{c}\big|_{z=H} n_{sy} + \left[\left(\overline{w} - v_s\right)\overline{c} - F\right]\big|_{z=H} n_{sz} = 0$$

No flux of sediment across water surface

- Introduce approximations (generally accurate for turbulent flow)

$$\int_0^H \overline{u}^2 dz \cong U^2 H\,,\quad \int_0^H \overline{v}^2 dz \cong V^2 H\,,\quad \int_0^H \overline{u}\,\overline{v}\,dz \cong UVH$$

ST. VENANT SHALLOW WATER EQUATIONS

Results for flow

$$\frac{\partial H}{\partial t} + \frac{\partial UH}{\partial x} + \frac{\partial VH}{\partial y} = 0$$

$$\frac{\partial UH}{\partial t} + \frac{\partial U^2 H}{\partial x} + \frac{\partial UVH}{\partial y} = -gH\left(\frac{\partial H}{\partial x} + \frac{\partial \eta}{\partial x}\right) - \frac{\tau_{bx}}{\rho}$$

$$\frac{\partial VH}{\partial t} + \frac{\partial UVH}{\partial x} + \frac{\partial V^2 H}{\partial y} = -gH\left(\frac{\partial H}{\partial y} + \frac{\partial \eta}{\partial y}\right) - \frac{\tau_{by}}{\rho}$$

Results for suspended sediment

$$\frac{\partial CH}{\partial t} + \frac{\partial q_{sx}}{\partial x} + \frac{\partial q_{sy}}{\partial y} = v_s\left(E - \bar{c}_b\right) \qquad q_{sx} = \int_0^H \overline{u}\,\overline{c}\,dz\,, \quad q_{sy} = \int_0^H \overline{v}\,\overline{c}\,dz$$

(q_{sx}, q_{sy}) denote volume suspended sediment transport rates per unit width in the (x, y) directions.

EXNER EQUATION OF CONSERVATION OF BED SEDIMENT AND EQUATION OF CONSERVATION OF SUSPENDED SEDIMENT

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_{bx}}{\partial x} - \frac{\partial q_{by}}{\partial y} + v_s(\bar{c}_b - E)$$

$$\frac{\partial CH}{\partial t} + \frac{\partial q_{sx}}{\partial x} + \frac{\partial q_{sy}}{\partial y} = v_s(E - \bar{c}_b)$$

Add:

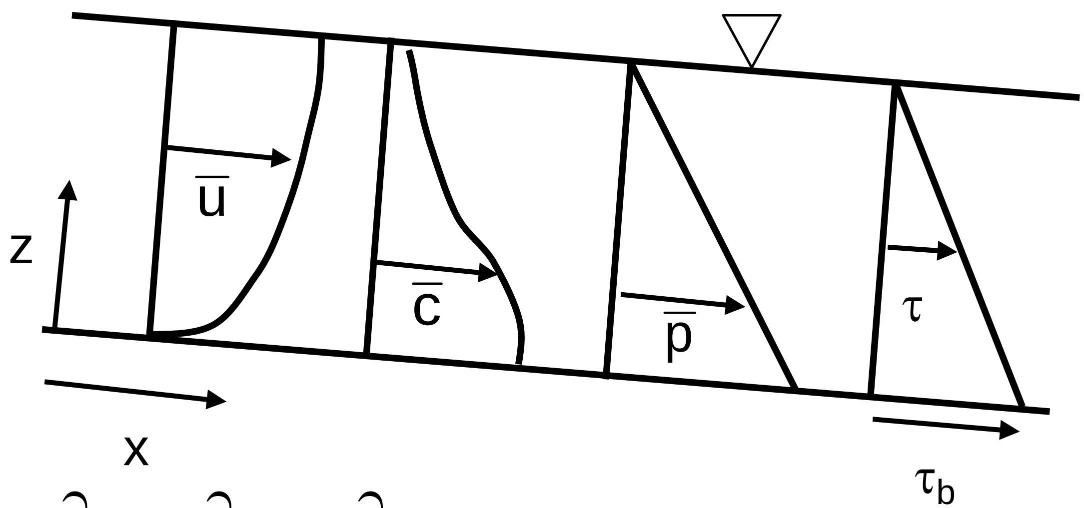
$$\frac{\partial}{\partial t}\left[(1-\lambda_p)\eta + \cancel{CH}\right] = -\frac{\partial q_{bx} + q_{sx}}{\partial x} - \frac{\partial q_{by} + q_{sy}}{\partial y}$$

Dilute suspensions

Formulation of sediment conservation in terms of total bed material load $\vec{q}_t$

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\frac{\partial q_{tx}}{\partial x} - \frac{\partial q_{ty}}{\partial y}\ , \quad q_{tx} = q_{bx} + q_{sx}\ , \quad q_{ty} = q_{by} + q_{sy}$$

STEADY, UNIFORM 1D OPEN-CHANNEL FLOW (NORMAL FLOW)



$$-\frac{\partial \eta}{\partial x} = S =$$

constant bed slope

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0\,, \quad \overline{w} = 0\,, \quad \tau_x \rightarrow \tau\,, \quad \tau_{bx} \rightarrow \tau_b = \rho u_*^2$$

$$u_*^2 = -\overline{u'w'}\Big|_{z=b}$$

Note that u_* provides a scale for the magnitude of near-bed turbulent velocity fluctuations.

STEADY, UNIFORM 1D OPEN-CHANNEL FLOW (NORMAL FLOW)

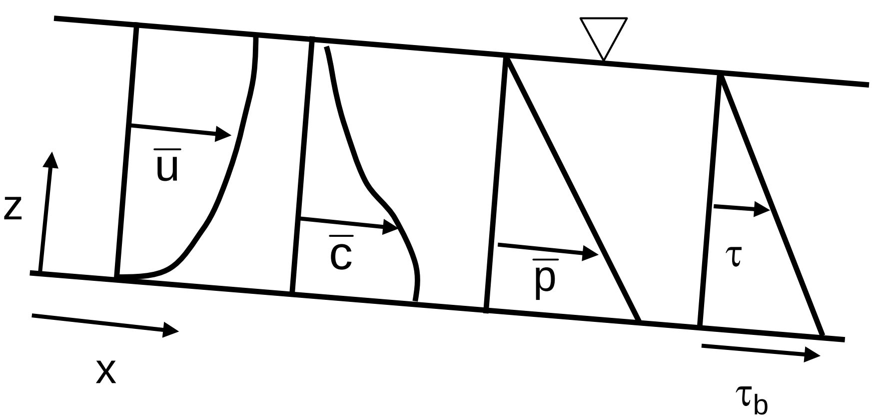
$$\cancel{\frac{\partial \bar{u}}{\partial t}} + \bar{u}\cancel{\frac{\partial \bar{u}}{\partial x}} + \bar{v}\cancel{\frac{\partial \bar{u}}{\partial y}} + \bar{w}\cancel{\frac{\partial \bar{u}}{\partial z}} = -g\cancel{\frac{\partial H}{\partial x}} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} - g\frac{\partial \eta}{\partial x}$$

$$\tau_{xz} \rightarrow \tau \qquad 0 = \frac{d\tau}{dz} + \rho g S$$

Solve for τ under boundary condition of vanishing τ at z = H:

$$\tau = \tau_b\left(1 - \frac{z}{H}\right) = \rho u_*^2\left(1 - \frac{z}{H}\right)$$

$$\tau_b \equiv \rho u_*^2 = \rho g H S$$



STEADY, UNIFORM 1D OPEN-CHANNEL FLOW

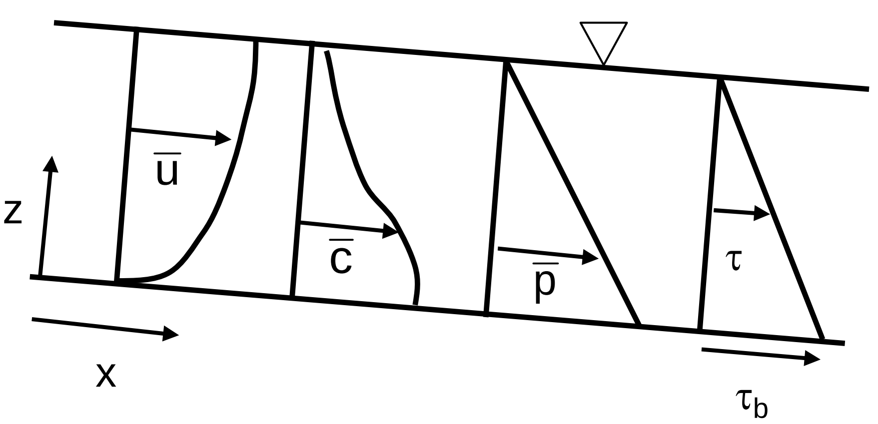
Close Reynolds stress with eddy viscosity assumption

$$\tau = -\rho\overline{u'w'} = \rho\nu_t\frac{d\bar{u}}{dz} \quad \text{or} \quad \rho\overline{u'w'} = -\nu_t\frac{d\rho\bar{u}}{dz}$$

Streamwise momentum is always diffused by turbulence from high concentration to low concentration. Note ν_t is a kinematic eddy viscosity with dimensions L^2T^{-1}.

Streamwise momentum balance:

$$\nu_t\frac{d\bar{u}}{dz} = u_*^2\left(1 - \frac{z}{H}\right)$$



PRANDTL MIXING LENGTH CLOSURE

$$\nu_t = \ell^2 \left| \frac{d\bar{u}}{dz} \right|$$

where ℓ is a length scale, so that $\nu_t \sim L^2T^{-1}$

Near a wall (i.e. sediment bed), $\ell = \kappa z$ where κ is an undetermined constant.

Consider zone where z/H << 1:

$$\nu_t \frac{d\bar{u}}{dz} = (\kappa z)^2 \left| \frac{d\bar{u}}{dz} \right| \frac{d\bar{u}}{dz} = u_*^2 \left(1 - \frac{z}{H}\right) \cong u_*^2 = \text{const}$$

But for flow under consideration, $\left| \frac{d\bar{u}}{dz} \right| \geq 0$

Thus $\frac{d\bar{u}}{dz} = \frac{u_*}{\kappa z}$ Integrate to get $\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ell n \left(\frac{z}{z_o} \right)$

where z_o corresponds to a constant of integration.

LOGARITHMIC LAW OF THE WALL FOR NEAR-BED STREAMWISE VELOCITY PROFILE

$$\frac{\overline{u}}{u_*} = \frac{1}{\kappa}\ell n\left(\frac{z}{z_o}\right)$$

Note that $\overline{u} \rightarrow -\infty$ as $z \rightarrow 0$.

The logarithmic law is not valid all the way to the bed. This is because very near the bed turbulence is damped, and the viscous shear stress, which has been neglected in the analysis, becomes dominant:

$$\tau = \rho\nu\frac{d\overline{u}}{dz} - \rho\overline{u'w'}$$

No problem: $\overline{u} = 0$ at $z = z_o$. So as long as $z_o/H \lll 1$, we can apply to an equivalent roughness height above the bed z_o.

LOGARITHMIC LAW OF THE WALL FOR NEAR-BED STREAMWISE VELOCITY PROFILE

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ell n\left(\frac{z}{z_o}\right) \qquad \kappa = 0.4$$

Karman constant evaluated from data

Note that $\bar{u} \to -\infty$ as $z \to 0$.
The logarithmic law is not valid all the way to the bed. This is because very near the bed turbulence is damped, and the viscous shear stress is neglected in the analysis:

$$\tau = \rho\nu\frac{d\bar{u}}{dz} - \rho\overline{u'w'}$$

Neglected in formulation *Drops out near bed*

No problem: $\bar{u} = 0$ at $z = z_o$. So as long as $z_o/H <<< 1$, we can apply to an equivalent roughness height above the bed z_o.

HYDRAULICALLY ROUGH LOGARITHMIC LAW OF THE WALL FOR FLOW OVER A GRANULAR BED

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ell n\left(\frac{z}{z_o}\right)$$

Nikuradse roughness height: $z_o = \frac{k_s}{30}$ k_s = equivalent sand grain roughness

Data from open-channel flow over granular beds: $k_s = n_k D_{s90}$

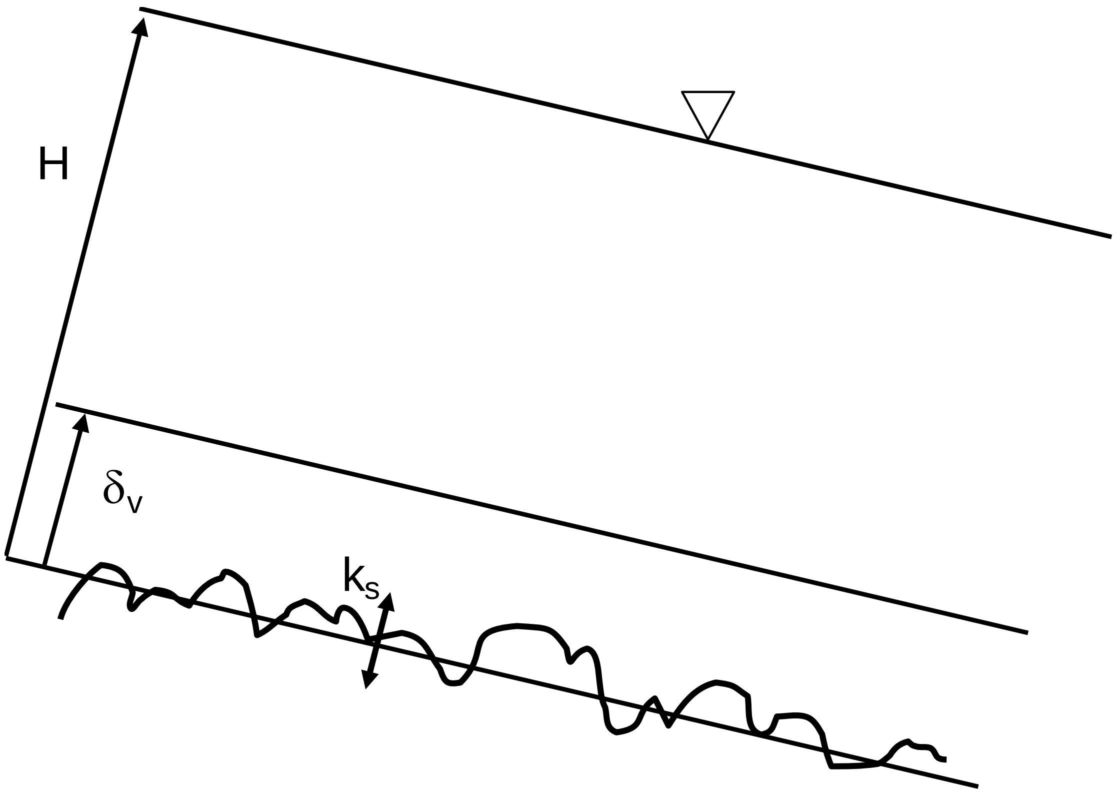
D_{s90} is a size such that 90% of the surface material is finer; n_k ~ 1.5 to 3 (about 2; Kamphuis, 1974).

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ell n\left(30\frac{z}{k_s}\right) = \frac{1}{\kappa}\ell n\left(\frac{z}{k_s}\right) + 8.5$$

GENERALIZATION TO INCLUDE VISCOUS EFFECTS

Height δ_v of the viscous sublayer:

$$\delta_v = 11.6 \frac{\nu}{u_*}$$



Hydraulically rough regime: neglect viscosity

$$\frac{k_s}{\delta_v} = \frac{1}{11.6} \frac{u_* k_s}{\nu} >> 1$$

Hydraulically smooth regime: neglect roughness height

$$\frac{k_s}{\delta_v} = \frac{1}{11.6} \frac{u_* k_s}{\nu} << 1$$

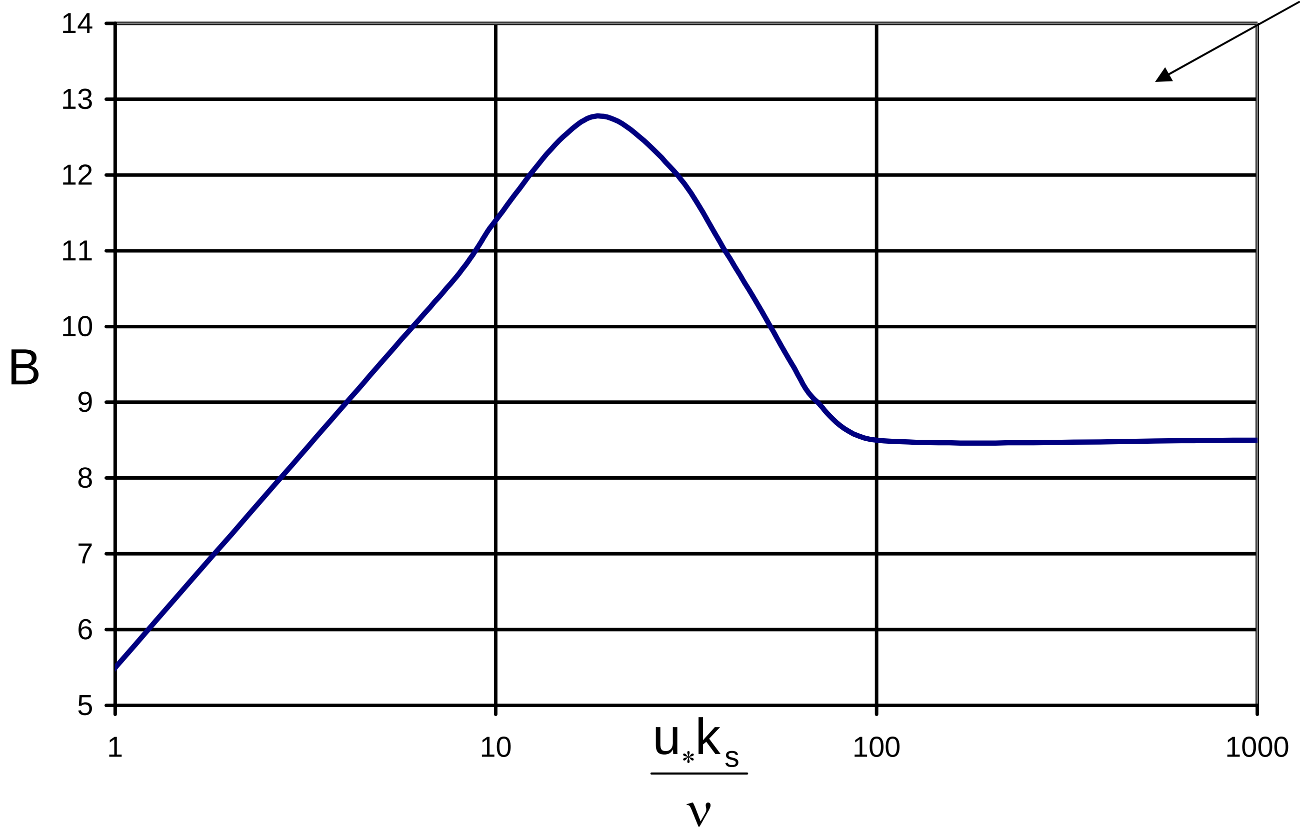
Transitional regime:

$$\frac{k_s}{\delta_v} = \frac{1}{11.6} \frac{u_* k_s}{\nu} = \text{order one}$$

GENERALIZED LOGARITHMIC LAW OF THE WALL

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ell n\left(\frac{z}{k_s}\right) + B\,, \quad B = B\left(\frac{u_* k_s}{\nu}\right)$$

Curve is something like this: see Schlichting (1968)



$$B = 8.5 \text{ for } \frac{u_* k_s}{\nu} > 70$$

$$B = 5.5 + \frac{1}{\kappa}\ell n\left(\frac{u_* k_s}{\nu}\right),$$

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ell n\left(\frac{u_* z}{\nu}\right) + 5.5$$

$$\text{for } \frac{u_* k_s}{\nu} < 2.5$$

GENERALIZATION OF LOG LAW AND EDDY VISCOSITY

The logarithmic law is strictly valid only close to the bed. In the case of steady, equilibrium open channel flow, however it does a generally good job of predicting the flow over the entire depth except very near the bed.

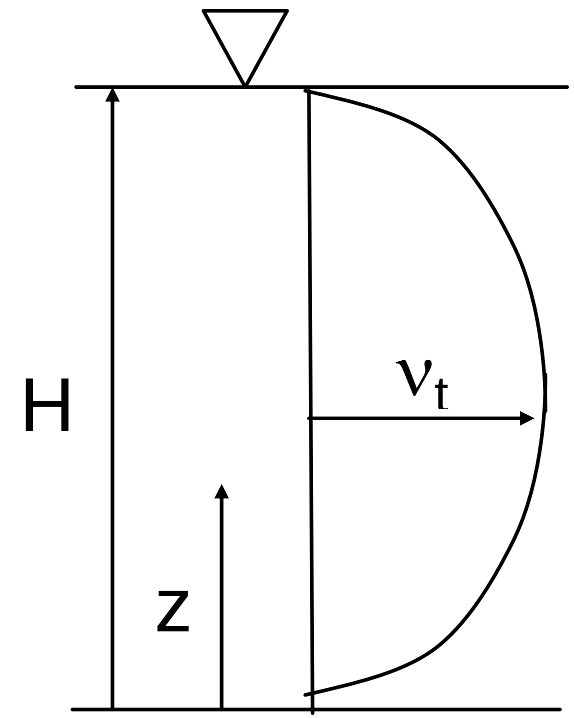
$$\frac{\overline{u}}{u_*} = \frac{1}{\kappa}\ell n\left(\frac{z}{k_s}\right) + B \qquad z \leq H$$

Momentum balance requires that:

$$\tau = \rho \nu_t \frac{d\overline{u}}{dz} = \rho u_*^2\left(1 - \frac{z}{H}\right)$$

Back-calculate the form of ν_t that yields the logarithmic law over the full depth:

$$\nu_t = \kappa u_* z\left(1 - \frac{z}{H}\right)$$



Parabolic form

KEULEGAN, MANNING-STRICKLER FRICTION RELATIONS

Depth-averaged velocity relation from rough logarithmic law:

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ell n\left(30\frac{z}{k_s}\right) \qquad UH = \int_0^H \bar{u}\,dz$$

Integrate to find Keulegan's friction law:

$$\frac{U}{u_*} = \frac{1}{\kappa}\ell n\left(11\frac{H}{k_s}\right) = \frac{1}{\kappa}\ell n\left(\frac{H}{k_x}\right) + 6$$

Manning-Strickler power approximation:

$$\frac{U}{u_*} = 8.1\left(\frac{H}{k_s}\right)^{1/6}$$

KEULEGAN, MANNING-STRICKLER FRICTION RELATIONS

Now by definition $$\tau_b = \rho u_*^2$$

Further defining a dimensionless bed friction coefficient C_f as

$$C_f = \frac{\tau_b}{\rho U^2}$$

The following friction relations are obtained:

Keulegan

$$C_f = \left[\frac{1}{\kappa}\ell n\left(11\frac{H}{k_s}\right)\right]^{-2}$$

Manning-Strickler

$$C_f = 0.0152\left(\frac{H}{k_s}\right)^{-1/3}$$

BOLD GENERALIZATION TO 2D, "MILDLY" DISEQUILIBRIUM FLOWS

$$\vec{\tau}_b = \left(\tau_{bx}, \tau_{by}\right), \quad \vec{U} = (U, V)$$

Assume that the bed shear stress vector is parallel to the depth-averaged velocity vector (often not true due to secondary flows), and that:

$$\left|\vec{\tau}_b\right| = \rho C_f \left|\vec{U}\right|^2$$

It follows that

$$\vec{\tau}_b = \rho C_f \left|\vec{U}\right| \vec{U}$$

Further assume that C_f can be estimated from the equilibrium Keulegan or Manning-Strickler forms.

CLOSURE FOR 2D ST. VENANT EQUATIONS

$$\left(\tau_{bx}, \tau_{by}\right) = \rho C_f \sqrt{\left(U^2 + V^2\right)}(U, V)$$

$$\frac{\partial H}{\partial t} + \frac{\partial UH}{\partial x} + \frac{\partial VH}{\partial y} = 0$$

$$\frac{\partial UH}{\partial t} + \frac{\partial U^2H}{\partial x} + \frac{\partial UVH}{\partial y} = -gH\left(\frac{\partial H}{\partial x} + \frac{\partial \eta}{\partial x}\right) - C_f \sqrt{\left(U^2 + V^2\right)}U$$

$$\frac{\partial VH}{\partial t} + \frac{\partial UVH}{\partial x} + \frac{\partial V^2H}{\partial y} = -gH\left(\frac{\partial H}{\partial y} + \frac{\partial \eta}{\partial y}\right) - C_f \sqrt{\left(U^2 + V^2\right)}V$$

In many applications to morphodynamics the above equations must be corrected for secondary flows that average to zero over the flow depth: e.g. secondary flow in river bends. Such a secondary flow can exert a bed shear stress, and cause sediment transport, in a direction not parallel to the depth-averaged velocity vector.