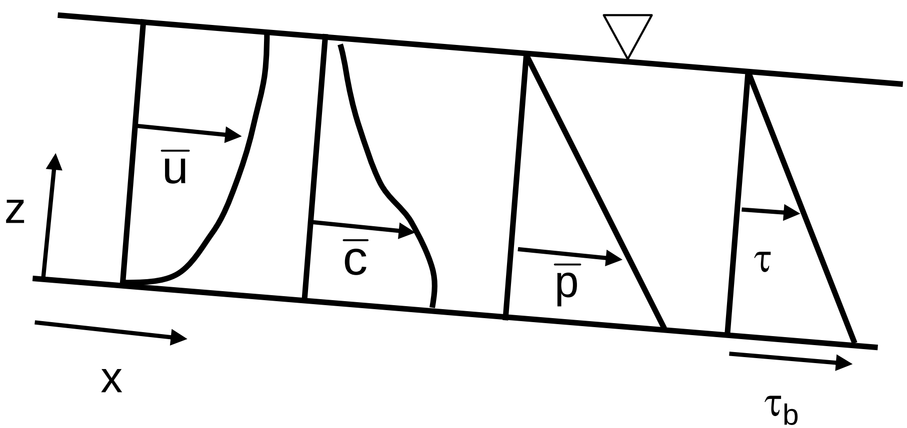
RELATIONS FOR ENTRAINMENT OF BED SEDIMENT INTO SUSPENSION

Consider the case of an equilibrium suspension in an equilibrium (normal) 1D open channel flow.

$$\cancel{\frac{\partial CH}{\partial t}} + \cancel{\frac{\partial q_{sx}}{\partial x}} + \cancel{\frac{\partial q_{sy}}{\partial y}} = v_s\left(E - \overline{c}_b\right)$$

$$\therefore \quad E = \overline{c}_b$$



Under equilibrium conditions the dimensionless entrainment rate E is equal to the near-bed average concentration of suspended sediment!

- Obtain empirical relation for E versus boundary shear stress for equilibrium conditions.
- With luck, the relation can be applied to conditions that are not too strongly disequilibrium.

A SMORGASBORD OF ENTRAINMENT RELATIONS

Smith & McLean (1977): Reference height b is at the top of the bedload layer (see their paper)

$$E = 0.65 \frac{\gamma_o \left(\frac{\tau^*_{smag}}{\tau^*_c} - 1 \right)}{1 + \gamma_o \left(\frac{\tau^*_{smag}}{\tau^*_c} - 1 \right)} \quad , \quad \gamma_o = 0.0024 \, , \quad \tau^*_{smag} = \left| \vec{\tau}^*_s \right| = \frac{\left| \vec{\tau}_{bs} \right|}{\rho R g D}$$

van Rijn (1984): Reference level b is a free variable (as long as b/H << 1)

$$E = 0.015 \frac{H}{b} \left(\frac{\tau^*_{smag}}{\tau^*_c} - 1 \right)^{1.5} \mathbf{Re}_p^{0.2} \, , \quad \mathbf{Re}_p = \frac{\sqrt{RgD}\, D}{\nu}$$

ENTRAINMENT SMORGASBORD contd.

Garcia and Parker (1991): reference height b = 0.05 H.

$$E = \frac{AZ_u^5}{1 + \frac{A}{0.3} Z_u^5} \quad , \quad Z_u = \frac{u_{*smag}}{v_s} \mathbf{Re}_p^{0.6} \, , \quad A = 1.3\text{x}10^{-7} \, , \quad u_{*smag} = \sqrt{\frac{\left|\vec{\tau}_{bs}\right|}{\rho}}$$

Wright and Parker (2003); corrects Garcia and Parker to cover large, low-slope streams: reference height b = 0.05 H.

$$E = \frac{AZ_u^5}{1 + \frac{A}{0.3} Z_u^5} \quad , \quad Z_u = \frac{u_{*smag}}{v_s} \mathbf{Re}_p^{0.6} S^{0.07} \, , \quad A = 5.7\text{x}10^{-7}$$

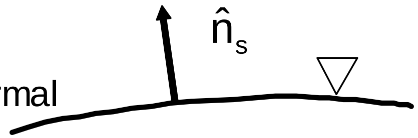
FORMULATION FOR SUSPENDED SEDIMENT

Reynolds-averaged equation of conservation of suspended sediment after applying slender flow approximations:

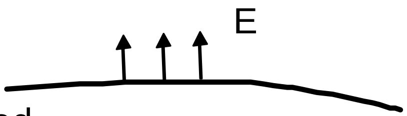
$$\frac{\partial \bar{c}}{\partial t} + \bar{u}\frac{\partial \bar{c}}{\partial x} + \bar{v}\frac{\partial \bar{c}}{\partial y} + \left(\bar{w} - v_s\right)\frac{\partial \bar{c}}{\partial z} = -\frac{\partial F}{\partial z}$$

$$F = \overline{c'w'}$$

no sediment flux normal to water surface



specified sediment entrainment rate at bed



Boundary condition at water surface: where $\hat{n}_s$ denotes a unit normal vector to the water surface,

$$\bar{u}\bar{c}\Big|_{z=H} n_{sx} + \bar{v}\bar{c}\Big|_{z=H} n_{sy} + \left[\left(\bar{w} - v_s\right)\bar{c} - F\right]_{z=H} n_{sz} = 0$$

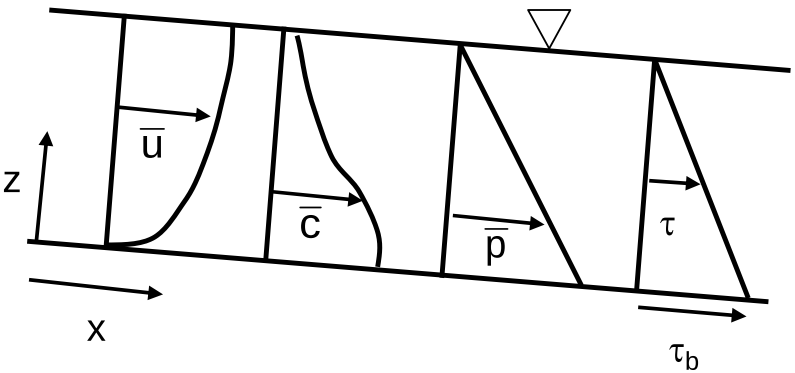
Boundary condition at bed: specified entrainment rate;

$$F\Big|_{z=b} = \overline{c'w'}\Big|_{z=b} = v_s E$$

CLOSURE FOR THE REYNOLDS FLUX

Again consider the case of equilibrium (normal) 1D open channel flow:

$$\cancel{\frac{\partial \bar{c}}{\partial t}} + \cancel{\bar{u}\frac{\partial \bar{c}}{\partial x}} + \cancel{\bar{v}\frac{\partial \bar{c}}{\partial y}} + \left(\cancel{\bar{w}} - v_s\right)\frac{\partial \bar{c}}{\partial z} = -\frac{\partial F}{\partial z}$$



$$\therefore \quad \frac{d}{dz}\left(F - v_s\bar{c}\right) = 0$$

$$\hat{n}_s = (0, 0, 1) \quad \therefore \quad \left.\left(F - v_s\bar{c}\right)\right|_{z=H} = 0$$

Integrate to get: $F - v_s\bar{c} = 0$ with bed b.c. $\left.F\right|_{z=b} = \left.\overline{c'w'}\right|_{z=b} = v_s E$

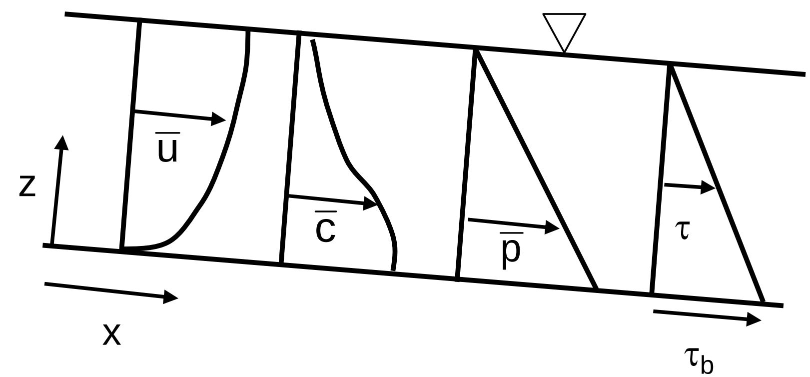
Closure based on Reynolds analogy

$$\rho\overline{u'w'} = -\nu_t \frac{d\rho\bar{u}}{dz} \quad \therefore \quad \overline{c'w'} = -\nu_t\frac{d\bar{c}}{dz} \qquad \nu_t = \kappa u_* z\left(1 - \frac{z}{H}\right)$$

THE ROUSE-VANONI SOLUTION FOR EQUILIBRIUM SUSPENDED SEDIMENT CONCENTRATION

$$\nu_t \frac{d\overline{c}}{dz} + v_s \overline{c} = 0 , \quad \nu_t = \kappa u_* z \left(1 - \frac{z}{H}\right)$$

$$\nu_t \left.\frac{d\overline{c}}{dz}\right|_{z=b} = v_s E$$



Integrate to get:

$$\frac{\overline{c}}{\overline{c}_b} = \left[\frac{(1-\varsigma)/\varsigma}{(1-\varsigma_b)/\varsigma_b}\right]^{\frac{v_s}{\kappa u_*}} , \quad \varsigma = \frac{z}{H} , \quad \varsigma_b = \frac{b}{H} , \quad \overline{c}_b = E$$

Rouse (1939)

AND NOW IT'S TIME FOR SPREADSHEET FUN!!

Go to RenesseRouseSpreadsheetFun.xls

Rouse-Vanoni Equilibrium Suspended Sediment Profile Calculator

$$\frac{\bar{c}}{\bar{c}_b} = \left[\frac{(1-\varsigma)/\varsigma}{(1-\varsigma_b)/\varsigma_b}\right]^{\frac{v_s}{\kappa u_*}} , \quad \varsigma = \frac{z}{H}, \quad \varsigma_b = \frac{b}{H}, \quad \bar{c}_b = E$$

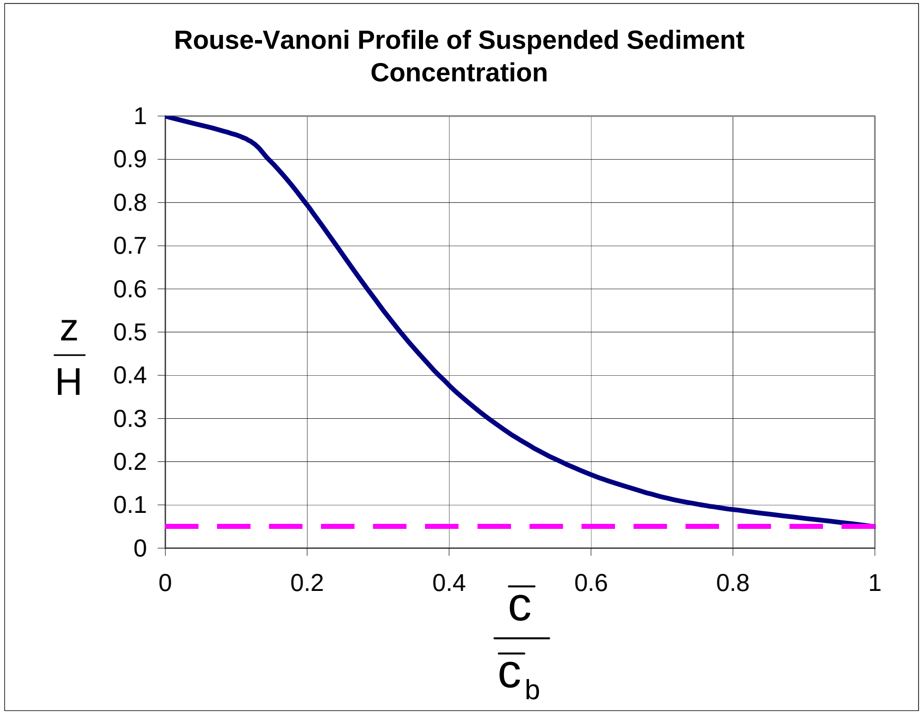
	Input	
b/H	0.05	
v_s	3	cm/s
u_*	0.2	m/s

u_*/v_s 6.6667

Sample Fall Velocities

D, mm	v_s, cm/s
0.0625	0.330
0.125	1.08
0.25	3.04
0.5	7.40
1	15.5
2	28.3

c/cb	z/H	ref
1	0.05	
0.755629	0.1	
0.63528	0.15	
0.557494	0.2	
0.500481	0.25	
0.455469	0.3	
0.418104	0.35	
0.385924	0.4	
0.357395	0.45	
0.331488	0.5	
0.307458	0.55	
0.28473	0.6	
0.262815	0.65	
0.241255	0.7	
0.219557	0.75	
0.197104	0.8	
0.172969	0.85	
0.145421	0.9	
0.109884	0.95	
0	1	
0		0.05
1		0.05



1D SUSPENDED SEDIMENT TRANSPORT RATE FROM EQUILIBRIUM SOLUTION

$$q_s = \int_0^H \overline{u}\,\overline{c}\,dz \cong \int_b^H \overline{u}\,\overline{c}\,dz \qquad \overline{c} = E\left[\frac{(1-\varsigma)/\varsigma}{(1-\varsigma_b)/\varsigma_b}\right]^{\frac{v_s}{\kappa u_*}}, \quad \varsigma = \frac{z}{H}, \quad \varsigma_b = \frac{b}{H}$$

Velocity profile over a rough bed: where k_c is a composite roughness height including the Nikuradse grain roughness k_s and the extra roughness effect of the bedforms,

$$\overline{u} = \frac{u_*}{\kappa}\ell n\left(30\frac{z}{k_c}\right) = u_*\left[\frac{1}{\kappa}\ell n\left(\frac{z}{k_c}\right) + 8.5\right]$$

If no bedforms,

$$k_c = k_s = n_k D_{s90}$$

If bedforms are present, evaluate the total friction coefficient $C_f = C_{fs}+C_{ff}$ and back-calculate k_c from

$$C_f = \left[\frac{1}{\kappa}\ell n\left(11\frac{H}{k_c}\right)\right]^{-2}$$

1D SUSPENDED SEDIMENT TRANSPORT RATE FROM EQUILIBRIUM SOLUTION contd.

$$q_s = \frac{u_* E H}{\kappa}\left\{\int_{\varsigma_b}^{1}\left[\frac{(1-\varsigma)/\varsigma}{(1-\varsigma_b)/\varsigma_b}\right]^{\frac{v_s}{\kappa u_*}} \ell n\left(30\frac{H}{k_c}\varsigma\right)d\varsigma\right\} \equiv \frac{u_* E H}{\kappa}\, I\left(\frac{u_*}{v_s}, \frac{H}{k_c}, \varsigma_b\right)$$

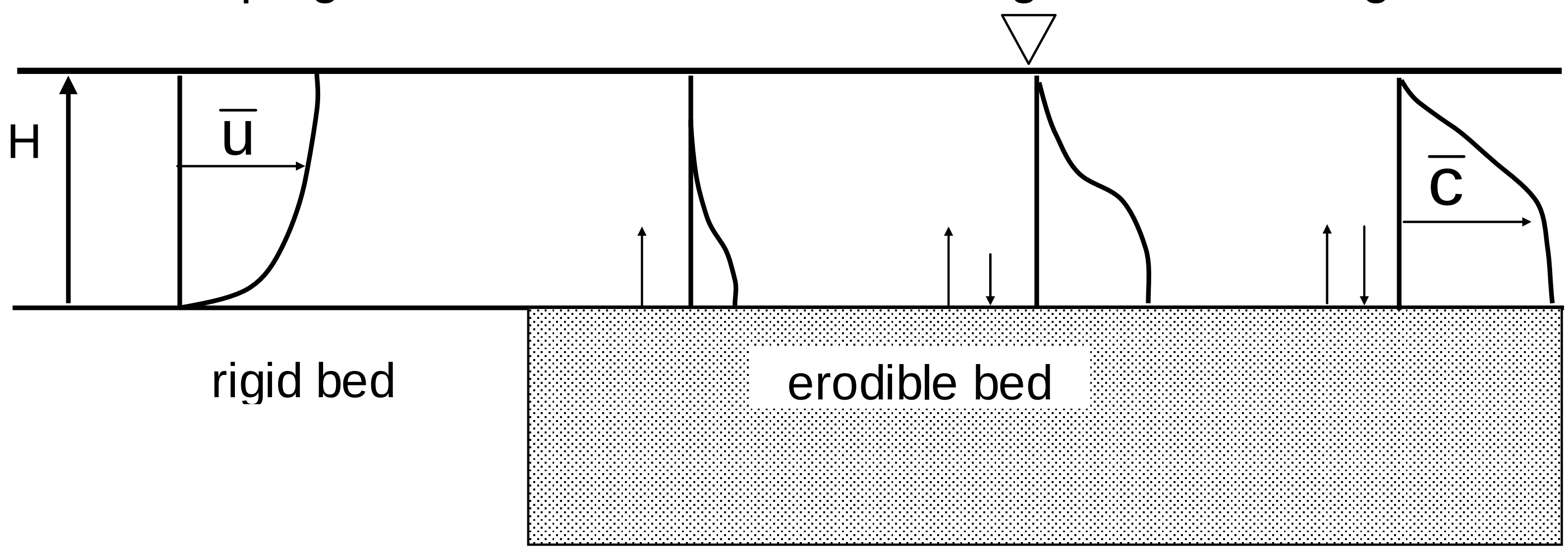
2D generalization for quasi-equilibrium flows:

$$\vec{q}_s = \frac{u_* E H}{\kappa}\frac{\vec{U}}{\left|\vec{U}\right|} I\left(\frac{u_*}{v_s}, \frac{H}{k_c}, \varsigma_b\right)$$

$$u_* = \sqrt{\frac{\left|\vec{\tau}_b\right|}{\rho}}$$

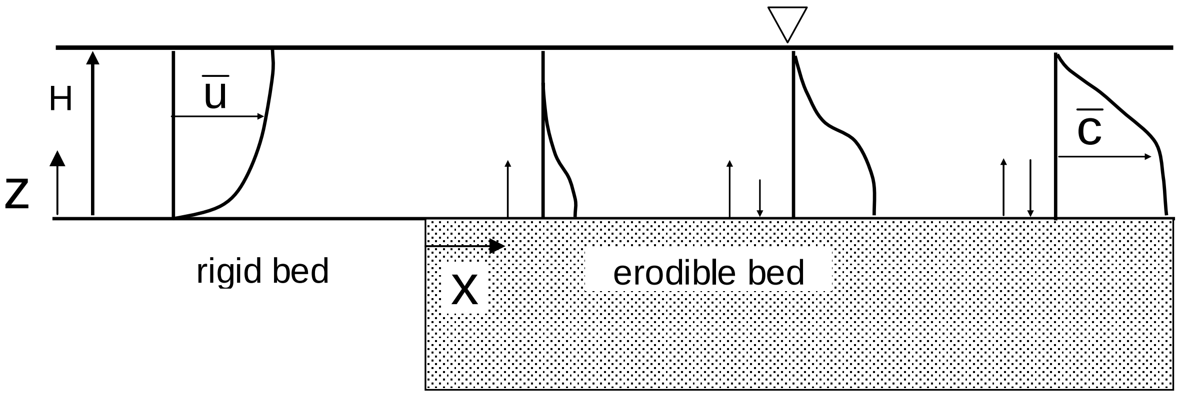
CLASSICAL CASE OF DISEQUILIBRIUM SUSPENSION: THE 1D PICKUP PROBLEM

Sediment-free equilibrium open-channel flow over a rough, non-erodible bed impinges on an erodible bed offering the same roughness.



- The flow can be considered quasi-steady over time spans shorter than that by which significant bed degradation occurs.
- The flow but not the suspended sediment profile can be considere to be at equilibrium.

THE 1D PICKUP PROBLEM contd.



governing eqn.

$$\cancel{\frac{\partial \bar{c}}{\partial t}} + \bar{u}\frac{\partial \bar{c}}{\partial x} + \cancel{\bar{v}\frac{\partial \bar{c}}{\partial y}} + \left(\cancel{\bar{w}} - v_s\right)\frac{\partial \bar{c}}{\partial z} = -\frac{\partial F}{\partial z} \quad \rightarrow \quad \bar{u}\frac{\partial \bar{c}}{\partial x} - v_s\frac{\partial \bar{c}}{\partial z} = \frac{\partial}{\partial z}\left(\nu_t \frac{\partial \bar{c}}{\partial z}\right)$$

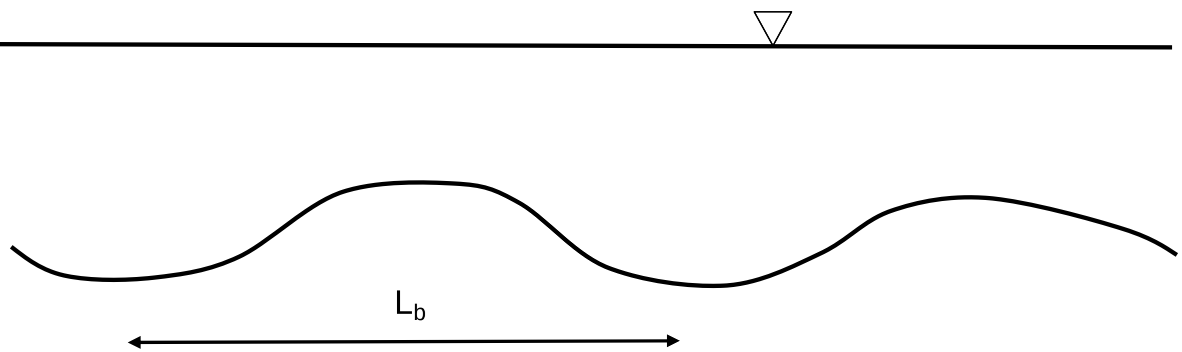
Boundary conditions

$$\left.\left(v_s\bar{c} + \nu_t\frac{\partial \bar{c}}{\partial z}\right)\right|_{z=H} = 0\,, \quad -\left(\nu_t\frac{\partial \bar{c}}{\partial z}\right)_{z=b} = v_s E\,, \quad \bar{c}\big|_{x=0} = 0$$

Solution yields the result that $\bar{c}(z, x) \rightarrow \bar{c}_{equil}(z) \quad \text{as} \quad x \rightarrow \infty$

Can be used to find adaptation length L_s for suspended sediment

WHICH VERSION OF THE EXNER EQUATION OF BED SEDIMENT CONTINUITY SHOULD BE USED FOR A MORPHODYNAMIC PROBLEM CONTROLLED BY SUSPENDED SEDIMENT?



Let L_b be the characteristic scale of the rhythmic feature to be explained, and L_s be the adaptation length for suspended sediment.

If $L_b \geq L_s$ use

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\vec{\nabla}\cdot\left(\vec{q}_b + \vec{q}_s\right)$$

where $\vec{q}_s$ is estimated from a quasi-equilibrium formulation:

$$\vec{q}_s = \frac{u_* E H}{\kappa}\frac{\vec{U}}{\left|\vec{U}\right|}\mathrm{I}\left(\frac{u_*}{v_s}, \frac{H}{k_c}, \zeta_b\right)$$

If $L_b < L_s$ use

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = -\vec{\nabla}\cdot\vec{q}_b + v_s\left(c_b - E\right)$$

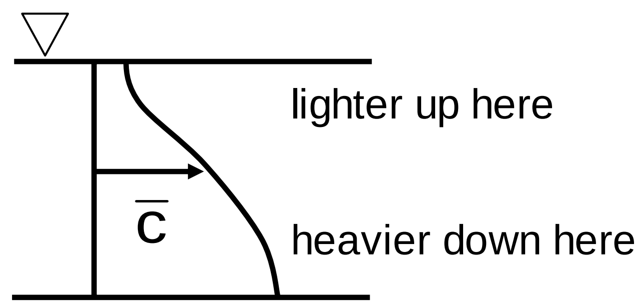
SELF-STRATIFICATION OF THE FLOW DUE TO SUSPENDED SEDIMENT

A flow is **stably stratified** if heavier fluid lies below lighter fluid. The density difference suppresses turbulent mixing.

Well, somewhere down there

The city of Phoenix, Arizona, USA during an atmospheric inversion

Sediment-laden flows are **self-stratifying**



$$\rho_{susp} = \rho(1-c) + \rho_s c = \rho(1+Rc)$$

$$\frac{\rho_{susp} - \rho}{\rho} = \varphi_e = Rc$$

Here φ_e = fractional excess density due to the presence of suspended sediment.

FLUX AND GRADIENT RICHARDSON NUMBERS

The damping of turbulence due to stable stratification is controlled by the **flux Richardson number Ri_f.**

$$Ri_f = \frac{g\overline{\varphi_e' w'}}{\left(-\overline{u'w'}\right)\frac{d\bar{u}}{dz}} = \frac{Rg\overline{c'w'}}{\left(-\overline{u'w'}\right)\frac{d\bar{u}}{dz}} =$$

[Rate of expenditure of turbulent kinetic energy in holding the (heavy) sediment in suspension]/[Rate of generation of turbulent kinetic energy by the flow]

Turbulence is not suppressed at all for $\mathbf{Ri_f}$ = 0. Turbulence is killed completely when $\mathbf{Ri_f}$ reaches a value near 0.2 (e.g. Mellor and Yamada, 1974)

Now let

$$\overline{u'w'} = -\nu_t \frac{d\bar{u}}{dz}, \quad \overline{c'w'} = -\nu_t \frac{d\bar{c}}{dz}$$

Then

$$Ri_g = \frac{-Rg\frac{d\bar{c}}{dz}}{\left(\frac{d\bar{u}}{dz}\right)^2} = Ri_f$$

where $\mathbf{Ri_g}$ denotes the **gradient Richardson Number**

SUSPENSION WITH SELF-STRATIFICATION

$$\nu_t = \nu_{to}\left(1 - 4.7 Ri_g\right) = \kappa u_* z\left(1 - \frac{z}{H}\right)\left[1 + 4.7\frac{Rg\dfrac{d\bar{c}}{dz}}{\left(\dfrac{d\bar{u}}{dz}\right)^2}\right]$$

Smith & McLean (1977)

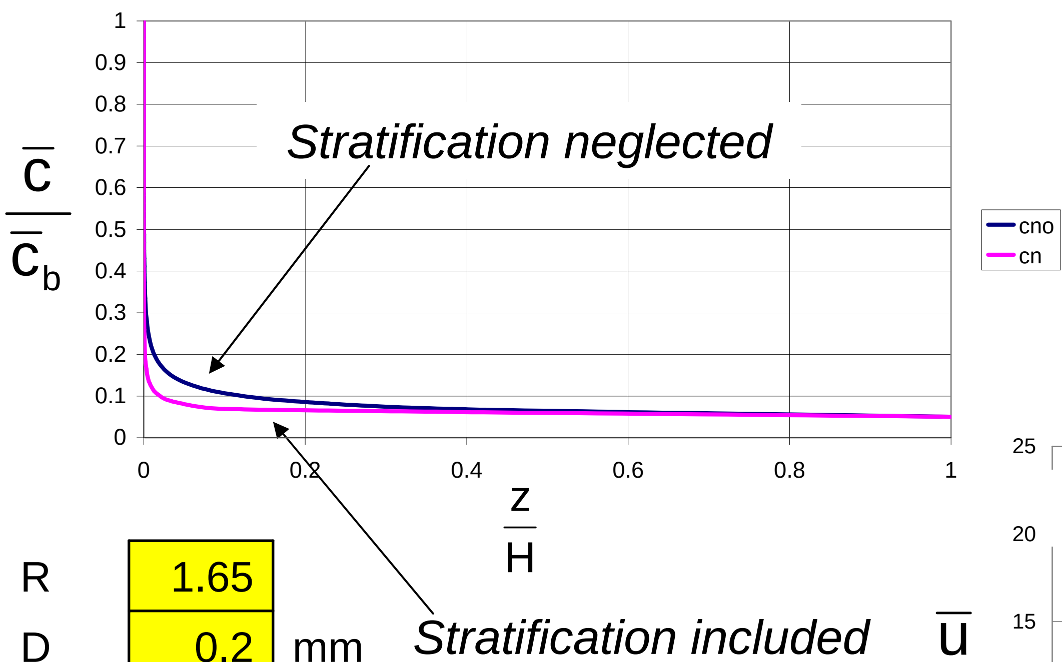
$$\nu_t \frac{d\bar{c}}{dz} + v_s \bar{c} = 0 \qquad -\nu_t \left.\frac{d\bar{c}}{dz}\right|_{z=b} = v_s E$$

$$\nu_t \frac{d\bar{u}}{dz} = u_*^2\left(1 - \frac{z}{H}\right) \qquad \frac{\left.\bar{u}\right|_{z=b}}{u_*} = \frac{1}{\kappa}\ln\left(30\frac{b}{k_c}\right)$$

Solve iteratively for concentration and velocity profiles in the presence of stratification.

SAMPLE CALCULATION (a) with Garcia-Parker entrainment relation

Effect of Density Stratification on Concentration Profiles
(cno - non-stratified, cn = stratified)

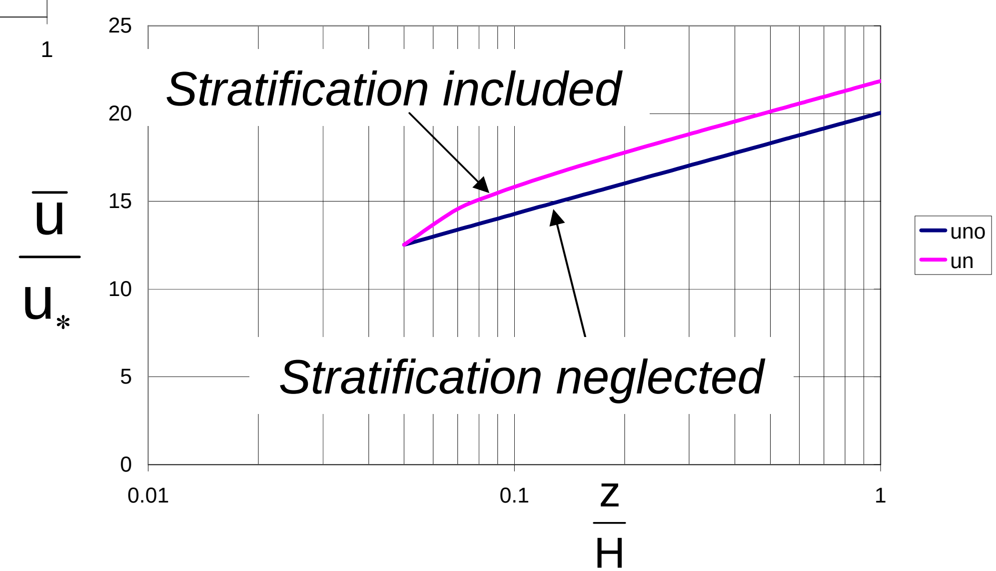


q_s with stratification = 0.70 x q_s without stratification

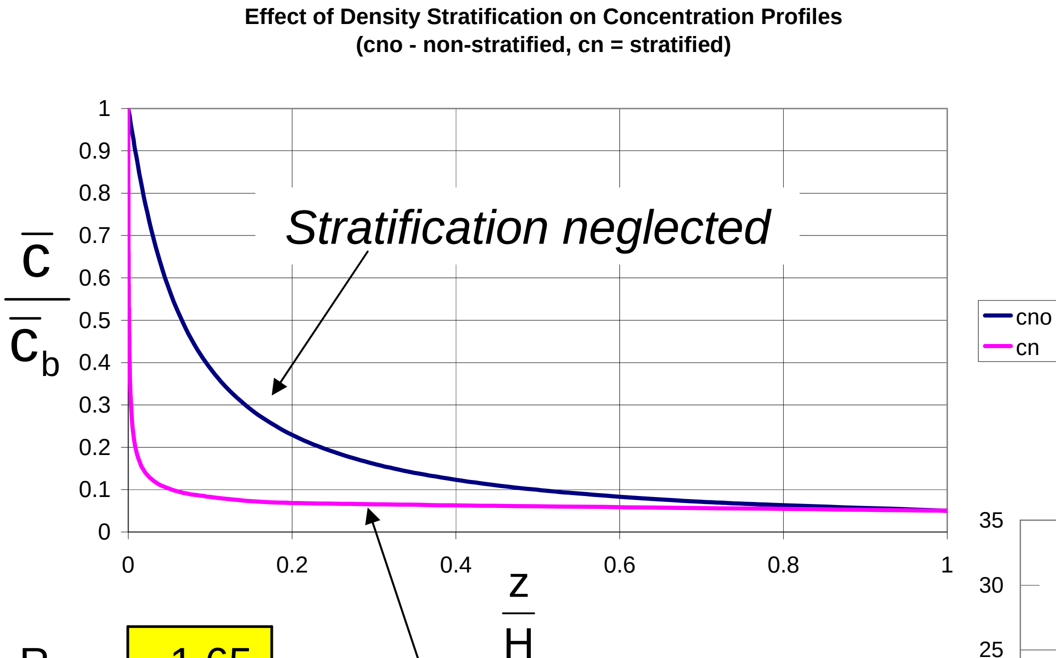
R	1.65	
D	0.2	mm
H	5	m
k_c	50	mm
ν	0.01	cm^2/s
u_*	**2**	cm

$$\bar{c}_b = 0.000115$$

Effect of Stratification on Velocity Profile
(uno = non-stratified, un = stratified)



SAMPLE CALCULATION (b) with Garcia-Parker entrainment relation



q_s with stratification = 0.28 x q_s without stratification

R	1.65	
D	0.2	mm
H	5	m
k_c	50	mm
ν	0.01	cm^2/s
u_*	**6**	cm

$$\bar{c}_b = 0.0255$$