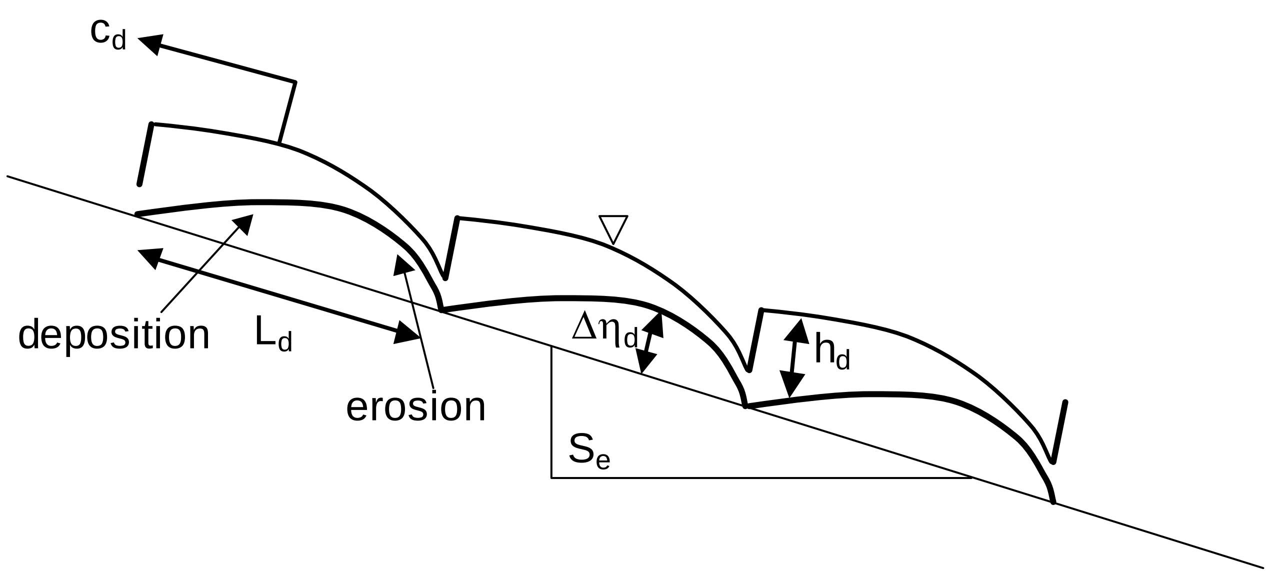
*Short Course: Geomorphological Fluid Mechanics*
Saint-Oyen, Italy, 2000

# CHAPTER 2
# TRANSPORTATIONAL CYCLIC STEPS




(view slides)

*Features*

- Dominant transport mode: suspended sediment
- Spatially periodic, or cyclic
- Each step bounded by hydraulic jump
- Froude-subcritical $\rightarrow$ supercritical down step
- Deposition upstream, erosion downstream
- Migrates upstream with constant speed $c_d$
- Steps maintain permanent form as they migrate

Question: is an equilibrium mobile-bed Froude-supercritical flow a *stable* equilibrium?

**St. Venant shallow water equations**
**Exner Equation of sediment continuity**
**Quasi-steady approximation**

$$u_d \frac{du_d}{dx_d} = -g\frac{dh_d}{dx_d} - g\frac{d\eta_d}{dx_d} - C_f u_d^2$$

$$u_d h_d = q_w$$

$$\frac{d}{dx_d}(q_w \chi_d) = v_s (E - r_o \chi_d)$$

$$(1-\lambda_p)\frac{\partial \eta_d}{\partial t_d} = v_s (r_o \chi_d - E)$$

Here "d" denotes "dimensioned"

$x_d$ = long coordinate — $t_d$ = time
$h_d$ = depth — $\eta_d$ = bed elevation
$u_d$ = flow velocity — g = grav. Accel.
$v_s$ = sed. fall velocity — $\lambda_p$ = bed porosity
$C_f$ = friction coefficient — $r_o = \chi_{db}/\chi_d$, o(1)

$q_w$ = water discharge/width (constant here)
$\chi_d$ = volume concentration of suspended sediment (dimensionless)

# “Constitutive” Relations

*Resistance*

$C_f$ = specified constant

*Entrainment into suspension*

$$E = \alpha_t [\tau_b^* - \tau_{th}^*]^n$$

$$\tau_b^* = \frac{\tau_b}{\rho R g D} \qquad \tau_b = \rho C_f u_d^2$$

$$\alpha_t = \alpha_t(\mathbf{Re}_p) \qquad \mathbf{Re}_p = \frac{\sqrt{RgD}\,D}{\nu}$$

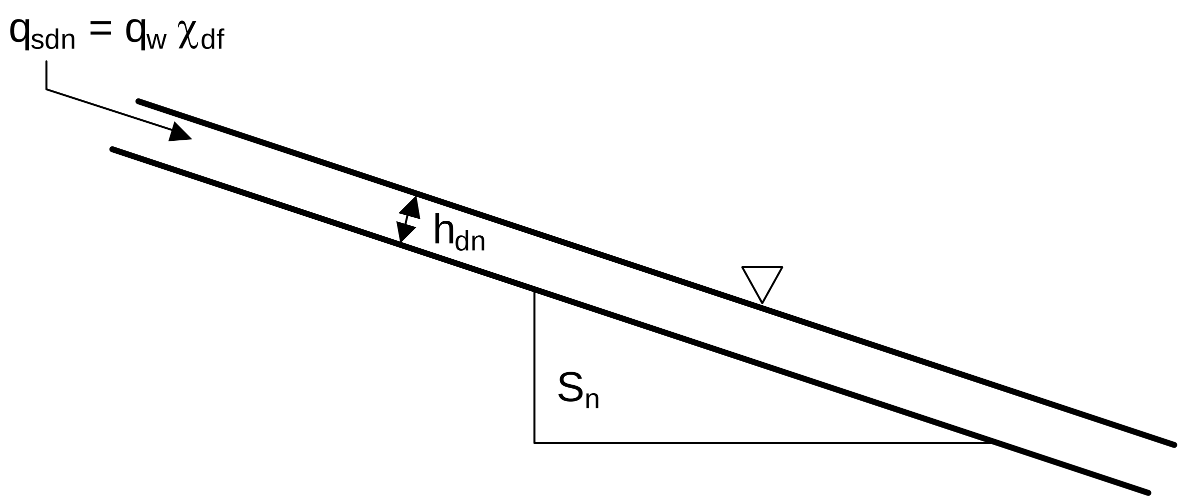
$$\tau_{th}^* = \frac{C_f u_{dt}^2}{RgD}$$

$u_{dt}$ = threshold velocity for bed erosion
D = grain size
$\nu$ = kinematic viscosity of water
$R = (\rho_s/\rho) - 1 \cong 1.65$ $\qquad$ $n \geq 1.5$

After some work,

$$E = \alpha_t \left(\tau_{th}^*\right)^2 \left[\left(\frac{u_d}{u_{dt}}\right)^2 - 1\right]^n$$

# Equilibrium mobile-bed state without steps




$q_w$ = constant

$q_{sdn}$ = feed sediment transport rate = $q_w \chi_{df}$ = constant

“n” = no steps

$$C_f u_{dn}^2 = g h_{dn} S_n$$

$$u_{dn} h_{dn} = q_w$$

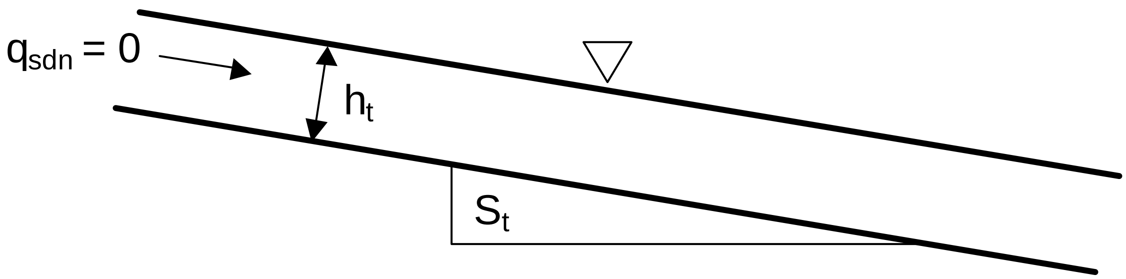
Assume thin flows, vigorous suspension $\therefore\ r_o \cong 1$

$$\chi_{dn} = E_n = \chi_{df} \qquad E_n = \alpha_t \left(\tau_{th}^*\right)^2 \left[\left(\frac{u_{dn}}{u_{dt}}\right)^2 - 1\right]^n$$

Specify $q_w$, $\chi_{df}$ ($R$, d, $C_f$, etc.): compute $u_{dn}$, $h_{dn}$, $S_n$

$$C_f \mathbf{Fr}_n^2 = S_n \qquad \mathbf{Fr}_n^2 = \frac{u_{dn}^3}{g q_w} > 1$$

# Equilibrium state at the threshold of bed erosion




$q_w$ = same as case of mobile-bed equilibrium

$q_{sdn} = \chi_{df} = 0$ (no sediment feed)

$$u_{dt} = \left( \frac{R g D \tau^{*}_{th}}{C_f} \right)^{1/2}$$

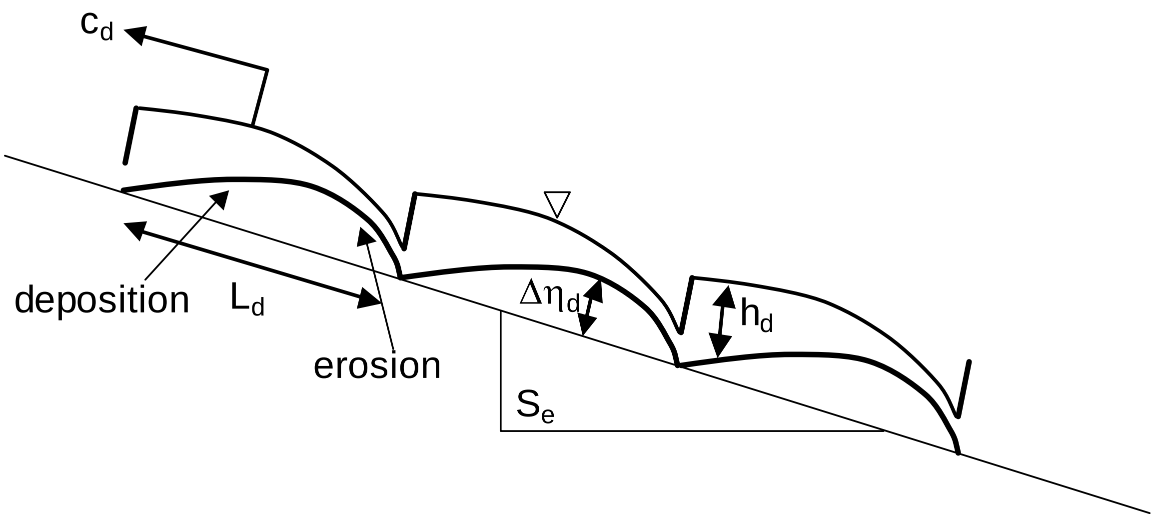
$$C_f u_{dt}^2 = g h_{dt} S_t$$
$$u_{dt} h_{dt} = q_w$$

Reduce to

$$C_f \mathbf{Fr}_t^2 = S_t \qquad \mathbf{Fr}_t^2 = \frac{u_{dt}^3}{g q_w}$$

Here we assume $\mathbf{Fr}_n > 1$, $\mathbf{Fr}_t < 1$

# Cyclic steps of permanent form




$c_d$ = wave speed (positive upstream)
$L_d$ = wave length
$S_e$ = equilibrium average slope with steps

In general,

$$\eta_d = \eta_{do} - S_e x_d + \eta_{ed}(x_d, t_d)$$

$$u_d = u_d(x_d, t_d)$$

$$h_d = h_d(x_d, t_d)$$

$$\chi_d = \chi_d(x_d, t_d)$$

Transform to

$$\tilde{x}_d = x_d + c_d t_d \qquad \tilde{t}_d = t_d$$

and drop dependency in $\tilde{t}_d$

**Result:**

Functional forms:

$$\eta_{ed} = \eta_{ed}(\tilde{x}_d) \qquad u_d = u_d(\tilde{x}_d)$$

$$h_d = h_d(\tilde{x}_d) \qquad \chi_d = \chi_d(\tilde{x}_d)$$

St: Venant equations

$$u_d \frac{du_d}{d\tilde{x}_d} = -g\frac{dh_d}{d\tilde{x}_d} + gS_e - g\frac{d\eta_{ed}}{d\tilde{x}_d} - C_f u_d^2$$

$$u_d h_d = q_w$$

$$q_w \frac{d\chi_d}{d\tilde{x}_d} = v_s(E - \chi_d)$$

Exner equation of bed sediment continuity

$$(1-\lambda_p)c_d \frac{d\eta_{ed}}{d\tilde{x}_d} = -v_s(E - \chi_d)$$

# Boundary conditions

Bed elevation

$$\eta_{ed}\Big|_{\tilde{x}_d=0} = 0 \qquad \eta_{ed}\Big|_{\tilde{x}_d=L_d} = 0$$

Flow velocity

$$u_d\Big|_{\tilde{x}_d=0} = u_{dt}$$

$$u_d\Big|_{\tilde{x}_d=L_d} = u_{dcon} \equiv \left(\frac{\sqrt{1+8\mathbf{Fr}_t^2}-1}{2}\right)^{-1} u_{dt}$$

Suspended sediment concentration

$$\frac{1}{T_d} q_w \int_0^{T_d} \chi_d dt_d = q_w \chi_{df} \qquad T_d = \frac{L_d}{c_d}$$

or thus

$$\frac{1}{L_d} \int_0^{L_d} \chi_d d\tilde{x}_d = \chi_{df}$$

## Non-dimensionalization

Introduce

$$u_d = u_{dt} u \qquad h_d = \frac{q_w}{u_{dt}} h \qquad \eta_{ed} = \frac{q_w}{u_{dt}} \eta$$

$$\tilde{x}_d = \frac{q_w}{u_{dt}} \frac{1}{S_t} x \qquad \chi_d = \chi_{df} \chi$$

Substitute into governing equations:

$$\frac{du}{dx} = \frac{S_r + \frac{\omega}{c}(E - \chi) - u^3}{\mathbf{Fr}_t^2 u - u^{-2}}$$

$$\frac{d\chi}{dx} = \omega(E - \chi)$$

$$c\frac{d\eta}{dx} = -\omega(E - \chi)$$

where

$$c = \frac{(1-\lambda_p)}{u_t \chi_f} c_d \qquad \omega = \frac{v_s}{S_t u_t} \qquad L = S_t \frac{u_t}{q_w} L_d$$

$$S_r = S_e / S_t$$

## Non-dimensionalization continued

$$\frac{du}{dx} = \frac{S_r + \frac{\omega}{c}(E - \chi) - u^3}{\mathbf{Fr}_t^2 u - u^{-2}}$$

$$\frac{d\chi}{dx} = \omega(E - \chi)$$

$$c\frac{d\eta}{dx} = -\omega(E - \chi)$$

where

$$E = \frac{(u^2 - 1)^n}{(u_n^2 - 1)^n}$$

Boundary conditions

$$\eta\big|_{x=0} = 0 \qquad \eta\big|_{x=L} = 0$$

$$u\big|_{x=0} = 1 \qquad u\big|_{x=L} = \left(\frac{\sqrt{1+8\mathbf{Fr}_t^2} - 1}{2}\right)^{-1}$$

$$L = \int_0^L \chi dx$$

# Reduction

Between

$$\frac{d\chi}{dx} = \omega(E - \chi) \qquad c\frac{d\eta}{dx} = -\omega(E - \chi)$$

find

$$\eta = \frac{1}{c}(\chi|_{x=0} - \chi)$$

Then apply

$$\eta|_{x=0} = 0 \qquad \eta|_{x=L} = 0$$

To get

$$\chi|_{x=L} = \chi|_{x=0}$$

# Final form of the problem

$$\frac{du}{dx} = \frac{S_r + \frac{\omega}{c}(E - \chi) - u^3}{\mathbf{Fr}_t^2 u - u^{-2}}$$

$$\frac{d\chi}{dx} = \omega(E - \chi)$$

where

$$E = \frac{(u^2 - 1)^n}{\left[\left(\frac{\mathbf{Fr}_n}{\mathbf{Fr}_t}\right)^{4/3} - 1\right]^n}$$

and

$$u\big|_{x=0} = 1 \qquad u\big|_{x=L} = \left(\frac{\sqrt{1+8\mathbf{Fr}_t^2} - 1}{2}\right)^{-1}$$

$$\chi\big|_{x=L} = \chi\big|_{x=0} \qquad L = \int_0^L \chi dx$$

# Character of the problem

*Overspecified system of ordinary differential equations*

2 first-order differential equations in u, $\chi$

+

4 parameters, $\mathbf{Fr}_t$, $\mathbf{Fr}_n$, $S_r$, c

*subject to*

4 boundary conditions

Thus if any two of $\mathbf{Fr}_t$, $\mathbf{Fr}_n$, $S_r$, c are specified, u, $\chi$ and the other two can be computed

But note the singularity in the denominator!

$$\frac{du}{dx} = \frac{S_r + \frac{\omega}{c}(E - \chi) - u^3}{\mathbf{Fr}_t^2 u - u^{-2}}$$

$$\mathbf{Fr}^2 = \frac{u_d^3}{g q_w} = \mathbf{Fr}_t^2 u^3$$

$$\therefore \mathbf{Fr} = 1 \text{ when } u = \mathbf{Fr}_t^{-2/3}$$

## Removal of singularity

Thus denominator is singular at $x = x_1$ where

$$u = u_1 \equiv \mathbf{Fr}_t^{-2/3}$$

Thus numerator of right-hand side of

$$\frac{du}{dx} = \frac{S_r + \frac{\omega}{c}(E - \chi) - u^3}{\mathbf{Fr}_t^2 u - u^{-2}}$$

must also vanish, yielding the constraint

$$\chi_1 = \frac{(\mathbf{Fr}_t^{-4/3} - 1)^n}{\left[\left(\frac{\mathbf{Fr}_n}{\mathbf{Fr}_t}\right)^{4/3} - 1\right]^n} - \frac{c}{\omega}(\mathbf{Fr}_t^{-2} - S_r)$$

at $x = x_1$

## Removal of singularity, continued

Now at $x = x_1$

$$\frac{du}{dx} = \frac{0}{0}$$

Applying L'hopital's rule,

$$3\mathbf{Fr}_t^2 (u_1')^2 + (3\mathbf{Fr}_t^{-4/3} - \frac{\omega}{c} E_{u1}) u_1' + \omega(\mathbf{Fr}_t^{-2} - S_r) = 0$$

where

$$u_1' = \left.\frac{du}{dx}\right|_{x_1}$$

$$E_{u1} = \left.\frac{dE}{du}\right|_{x_1} = \frac{n\mathbf{Fr}_t^{-2/3}(\mathbf{Fr}_t^{-4/3} - 1)^{n-1}}{\left[\left(\frac{\mathbf{Fr}_n}{\mathbf{Fr}_t}\right)^{4/3} - 1\right]^n}$$

## Solution procedure

Specify all parameters such as $\tau_{th}^*$, $\omega$, $C_f$, etc.

Specify values of $\mathbf{Fr}_n$ and $\mathbf{Fr}_t$.

Guess values of $S_r$, c

Define the variable $x_r = x - x_1$.

Integrate upstream from $x_r = 0$ in u and $\chi$ until

$$x_r = -L_u$$

where

$$u = 1 \quad \text{(threshold conditions)}$$

Integrate downstream from $x_r = 0$ in u and $\chi$ until

$$x_r = L_d$$

where

$$u = \left( \frac{\sqrt{1 + 8\mathbf{Fr}_t^2} - 1}{2} \right)^{-1}$$

For each $S_r$, c thus get

$$L = L_u + L_d = L(c, S_r)$$

$$u = u(x; c, S_r) \quad \chi = \chi(x; c, S_r)$$

## Solution procedure, continued

Write remaining two boundary conditions in the form

$$F_1(c, S_r) \equiv$$
$$\chi[L(c, S_r); c, S_r] - \chi(0; c, S_r) = 0$$

$$F_2(c, S_r) \equiv$$
$$L(c, S_r) - \int_0^{L(c, S_r)} \chi(x; c, S_r) dx = 0$$

Use bisection or Newton-Raphson technique to obtain improved guesses of c and $S_r$ until convergence

Sample case:

Experiments using 19 micron, 45 micron and 120 micron silica flour

Depths are 1.1 to 7.0 mm

Flow before steps form almost always supercritical: $\mathbf{Fr}_n > 1$