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SOMEWHAT LESS RANDOM NOTES ON BEDROCK INCISION

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Mark II

Discussion with: Greg Tucker, Alan Howard, Rudy Slingerland, Brad Murray, John Swenson, Tom Drake, Bill Dietrich, Kelin Whipple, Leigh Royden, Leonard Sklar and Ben Crosby

PREFACE

The notes below were prepared by the author on March 21, 2002, while the author was in residence as the Crosby Lecture in the Department of Earth, Atmospheric and Planetary Science at the Massachusetts Institute of Technology. They are called “Mark II” because a first version was prepared and distributed on February 22, 2002. The only changes that have been made in the present version is that references are properly specified and given.

INTRODUCTION

The notes presented here are an extension of notes prepared on February 22, 2002. They have benefited from input from Rudy Slingerland, Leonard Sklar and Bill Dietrich in regard to the formulation for wear, and input from Kelin Whipple and Ben Crosby in regard to macroabrasion. In addition, a paper by Alan Howard proved useful.

This document is somewhat rambling, so in order to get reactions and inputs from busy people I would like to highlight some tentative conclusions.

- To date most work on river profiles in incisional systems have looked at slope S versus area A. The analysis here suggests that the natural relation is in terms of S versus the parameter $\chi = A/B_c$, where B_c denotes channel width. Note that χ has the dimensions of length.
- The analysis suggests that macroabrasion, here defined as enhanced breakup by cracking of bedrock due to the collision of bedload particles, cannot simply be wrapped up into wear.
- The character of the problem of the time development of long profiles when steady state does not prevail changes dramatically from previous treatments based on the “standard” (m&n’s) model (outlined below). In particular, the incision rate at a given point becomes a function of the incision rate at every point upstream. This is because the incision due to wear and macroabrasion at a given point is driven by tools, the abundance of which is governed by incision rates upstream. This makes the problem tricker to solve, but also open up the possibility for a rich range of behavior.

References are still somewhat sparse here. I'm gradually compiling a list of the relevant references. Any and all comments and input are welcomed.

THE "STANDARD MODEL"

The Exner equation of sediment conservation for an incisional stream takes the form

$$(1-\lambda_p)\frac{\partial\eta}{\partial t} = \upsilon - v_I \qquad (1)$$

where η denotes bed elevation, t denotes time, υ = uplift rate ([L/T] where L denotes dimension of length, T denotes dimension of time), λ_p denotes bedrock porosity (if any) and v_I denotes the incision rate (L/T). The standard formulation for v_I is

$$v_I = aA^m S^n \qquad (2)$$

where A(x) denotes the drainage basin area upstream of the point at distance x from a virtual origin near the headwater of the main-stem stream, and

$$S = -\frac{\partial\eta}{\partial x} \qquad (3)$$

denotes bed slope. The above formula (2) has served a useful purpose to date, but has at least two problems.

The first of these concerns dimensions. The dimensions of the parameter a in (2) are [L^{1-2m}/T], where m is a function of the author of the paper in question. I can't think about physical mechanisms in terms of dimensions that float around as free parameters.

The second concerns the almost inevitable transition from bedrock to alluvium. Many studies have been done using (1) and (2) to study steady-state incision with constant uplift υ. Yet in general the bedrock reach has to end somewhere, and after that point it is impossible to realize a steady state, because incision cannot balance uplift in an alluvial reach.

ALTERNATIVE MODEL

The goal is to a) make specific statements about mechanisms, and b) delineate simple parameters with sensible dimensions that offer a hope of quantifying the problem based on field data, and to a certain extent experiments. There are undoubtedly many ways to cause a river to incise into its own bedrock. Here two are considered; wear and plucking. Wear is treated in terms of relations of the same status as those used to predict gravel abrasion in rivers, leading to a coefficient β_w having the dimensions of [1/L]. The stones that do the wear are assumed to have a characteristic size D_w.

Plucking is conceived as follows. There is an "aging layer" just below the surface of the bedrock of thickness L_a in which the bedrock is gradually becoming fractured and loosened over time by a) stress release within the rock layer, b) repeated wetting and drying in the case of a badlands bedrock, c) bioturbation of the surface layer in the case of weakly consolidated mud bedrock or d) surface chemical effects. The process of fracturing leads to the formation of chunks that can be plucked and transported out. The process by which an unfractured aging layer of thickness L_a (dimension L) becomes fractured to pluckable sizes is governed by an "aging time" T_{pa} (dimension [T]) which loosely corresponds to a half-life for fracturing in the aging layer.

This "aging" layer is capped by a "battering layer" of thickness D_p, where D_p is a characteristic size of pluckable stone. Within the "battering layer" the process of fracturing is abetted by wear particles that collide with the surface of the "battering layer". Thus the wear particle act to increase the natural frequency of fracturing (macroabrasion), and thus the production of pluckable particles, in addition to grinding away the bedrock to sand or silt (pure wear).

For simplicity the model assumes just two sediment sizes, i.e. a size D_w that actually does the wearing and abets the fracturing, and a size $D_p > D_w$ for the characteric dimension of a fractured piece of bedrock that can be plucked out and removed. In addition to these two sizes, it is assumed that there is a "wash load" of finer material that is ineffective as an agent of either wear or macroabrasion. For example, the sand and silt might be treated as "wash load" in regard to bedrock incision, D_w might be a typical gravel size D_p might be equivalent to a cobble size. The model can presumably be generalized for full grain size distributions.

Presumably no incision whatsoever is occurring in bedrock streams for the great majority of time. With this in mind (1) is modified to

$$(1-\lambda_p)\frac{\partial \eta}{\partial t} = \upsilon - \mathrm{I} v_I \qquad (4a)$$

where I denotes an intermittency associated with the (probably episodic) events that do most of the incision (Paola et al., 1992), and v_I denotes the incision rate during those events. The incision rate is further decomposed as follows;

$$v_I = v_{Iw} + v_{Ip} \qquad (4b)$$

where v_{Iw} corresponds to incision due to wear and v_{Ip} corresponds to incision due to plucking.

SUBMODEL FOR WEAR

Let q(x) denote the volume transport rate of sediment in the stream per unit width (L^2/T) during the storm events that drive abrasion. Let the fraction of this load that consists of particles coarse enough to do the wear be α. The volume transport rate per unit width q_w of sediment coarse enough to wear the bedrock is then given as

$$q_w = \alpha q \tag{5}$$

In a first bonehead formulation, α might be set simply equal to the fraction of the load that is gravel or finer. A more sophisticated formulation might use a discriminator such as the ratio of shear velocity to fall velocity, as Sklar (thesis) has done.

Consider the case of saltating bedload particles. Let E_{saltw} denote the volume rate at which saltating wear particles bounce off the bed per unit bed area [L/T] and L_{saltw} denotes the characteristic saltation length of wear particles [L]. It follows from simple continuity that

$$q_w = E_{saltw} L_{saltw} \tag{6}$$

The mean number of bed strikes by wear particles per unit bed area per unit time is equal to E_{saltw}/V_w, where V_w denotes the volume of a wear particle. It is assumed that with each collision a fraction r of the particle volume is ground off the bed (and a commensurate, but not necessary equal amount is ground off the wear particle). The rate of bed incision v_{lw} due to wear is then given as (number of strikes per unit bed area) x (volume removed per unit strike), or

$$v_{lw} = \frac{E_{saltw}}{V_w} r V_w \tag{7}$$

Reducing (7) with (6), it is found that

$$v_{lw} = \beta_w q_w \quad , \quad \beta_w = \frac{r}{L_{saltw}} \tag{8a,b}$$

Here the parameter β_w has the dimensions [1/L], and has exactly the same status as the abrasion coefficients used to study downstream fining by abrasion in rivers (e.g. Parker, 1991). Both Sklar and Dietrich (1997) and Slingerland et al. (1997) have already presented forms of the type of (8a), but the recognition of the status of the coefficient as a standard abrasion coefficient appears to be new.

In a first model of wear β_w could be treated as a constant. In so far as L_{saltw} depends on flow conditions and r depends on rock type and perhaps the strength of the collision, β_w might be expected to vary somewhat with flow and lithology. Sklar (thesis) has presented a relation for this variation.

Equation (8a) is valid only to the extent that all wear particles collide with exposed bedrock. If wear particles partially cover the bed, the wear rate should be commensurately reduced. This effect can be quantified in terms of the ratio q_{wear}/q_{wearc}, where q_{wearc} denotes the capacity transport rate of wear particles. Let p_o denote the areal fraction of surface bedrock that is not covered with wear particles. In general p_o can be expected to approach unity as $q_{wear}/q_{wearc} \rightarrow 0$, and approach zero as $q_{wear}/q_{wearc} \rightarrow 1$. Various authors, an in particular Sklar and Dietrich (1997), Slingerland et al. (1997), Sklar and Dietrich (1998) and Sklar (thesis) have proposed the form

$$p_o = \left(1 - \frac{q_{wear}}{q_{wearc}}\right)^{n_o} \tag{9}$$

With this in mind (8a) is modified to

$$v_{lw} = \beta_w q_w p_o \tag{10}$$

The simplest choice for n_w is unity, or

$$p_o = 1 - \frac{q_w}{q_{wc}} \tag{11}$$

The parameter q_{wearc} can be quantified in terms of standard bedload transport relations. A generalized relation of the form of Meyer-Peter and Müller (1948), for example, takes the form

$$q_{wearc} = \sqrt{RgD_w}\,D_w \gamma_T \left(\frac{\tau_b}{\rho RgD_w} - \tau_c^*\right)^{n_T} \tag{12}$$

where g denotes the acceleration of gravity, ρ denotes water density, τ_b denotes bed shear stress, $R = (\rho_s/\rho) - 1$ where ρ_s denotes sediment density, τ_c^* denotes a dimensionless critical Shields number, γ_T is a dimensionless constant and n_T is a dimensionless exponent. For example, in the implementation of Fernandez Luque and van Beek (1976), $\gamma_T = 8$, $n_T = 1.5$ and τ_c^* is between 0.03 and 0.045.

Summarizing, then, the relation for incision due to wear is

$$v_{lw} = \beta_w \alpha q \left(1 - \frac{\alpha q}{q_{wearc}}\right)^{n_o} \tag{13}$$

Note that v_{lw} drops to zero when αq becomes equal to q_{wc}, downstream of which a completely alluvial gravel-bed stream is found. That is, the above formulation can describe the end point of the incisional zone as well as the incision rate.

SUBMODEL FOR SEDIMENT TRANSPORT RATE

A routing model is necessary to determine q, and thus q_{wear}. Let $B_c(x)$ denote channel width. The equation of sediment conservation on a bedrock reach can be written as

$$\frac{d}{dx}(qB_c) = q_h \tag{14}$$

where q_h denotes the volume of sediment per stream length per unit time entering the channel from the hillslopes (either directly or through the intermediary of tributaries). Several models can be postulated for q_h depending on hillslope dynamics. For the sake of a simple example, it is assumed that the watershed consists of easily-weathered rocks that are rapidly uplifted, so that bed lowering by channel incision results in hillslope lowering at the same rate. In this case

$$q_h = v_l \frac{dA}{dx} \tag{15}$$

as illustrated in Figure 1. Note that in (15) v_l is the total incision rate and not just that due to wear. And remember, (15) is just an example, a stand-in that must later be generalized to forms including e.g. hillslope diffusion, hillslope relaxation due to landslides driven by e.g. earthquakes or saturation in the absence of uplift etc.

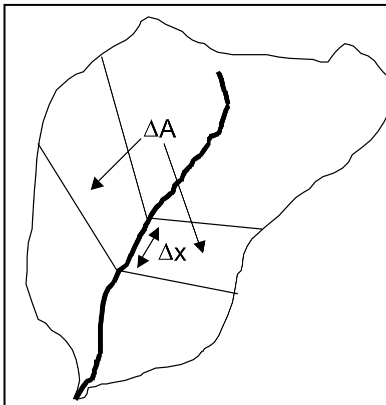


Figure 1

Between (14) and (15) it is found that

$$\frac{d}{dx}(qB_c) = v_l \frac{dA}{dx} \tag{16}$$

In the case of steady-state incision in response to spatially uniform (piston-style) uplift (16) reduces to the form

$$q = v_I \chi \quad , \quad \chi = \frac{A}{B_c} \tag{17a,b}$$

The above is the first of many relations in which the combination χ = A/B_c enters naturally into the problem.

To obtain an approximate treatment of the case of deviation from steady-state incision in response to piston-style uplift, it is useful to postulate the structure relation

$$B_c = \alpha_b A^{n_b} \tag{18a}$$

or equivalently

$$B_c = \tilde{\alpha}_b \chi^{m_b} \ , \quad \tilde{\alpha}_b = \alpha_b^{1/(1-n_b)} \ , \quad m_b = \frac{n_b}{1-n_b} \tag{18b}$$

Between (16) and (18b) it is possible to obtain the solution

$$q = \frac{m_b + 1}{\chi^{m_b}} \int_0^{\chi} v_I \chi'^{m_b} d\chi' \tag{19}$$

Combining (13) and (19), it is found that

$$v_{Iw} = \beta_w \alpha \left(\frac{m_b + 1}{\chi^{m_b}} \int_0^{\chi} v_I \chi'^{m_b} d\chi' \right) \left[1 - \frac{\alpha \left(\frac{m_b + 1}{\chi^{m_b}} \int_0^{\chi} v_I \chi'^{m_b} d\chi' \right)}{q_{wearc}} \right]^{n_o} \tag{20}$$

Note that the form of (20) is such that the incision rate due to wear v_{Iw} at a point is a function of the channel incision rate v_I (due to all mechanisms) at every point upstream. While (16) is only a sample relation for sediment yield, alternative formulations can be expected to yields results with the same structure.

A relation of the same type as (19a) can be obtained by assuming a power relation between B_c and flood discharge Q_f.

SUBMODEL FOR BOUNDARY SHEAR STRESS

The standard formulation for boundary shear stress places it proportional to the square of the flow velocity U = q_f/H where q_f denotes the flow discharge per unit width and H denotes flow depth. More precisely,

$$\tau_b = \rho C_f \frac{q_f^2}{H^2} \tag{21}$$

where C_f is a friction coefficient, here assumed constant for simplicity. For the steep slopes of bedrock streams the normal flow approximation, according to which the downstream pull of gravity just balances the resistive force at the bed, should apply, so that momentum balance takes the form

$$\tau_b = \rho C_f \frac{q_f^2}{H^2} = \rho g H S \tag{22}$$

Solving for H in terms of S and substituting in for the boundary shear stress, it is found that

$$\tau_b = \rho C_f^{1/3} g^{2/3} q_f^{2/3} S^{2/3} \tag{23}$$

Now let i denote the precipitation rate (L/T). Assuming no storage of water in the basin, the balance for water flow is

$$q_f B_c = iA \tag{24}$$

Between (12), (23) and (24), then, the capacity bedload transport rate of effective tools for wear is given as

$$q_{wearc} = \sqrt{RgD_w} D_w q_{wearc}^*$$
$$q_{wearc}^* = \gamma_T \left[\frac{C_f^{1/3} i^{2/3} \chi^{2/3}}{R g^{1/3} D_w} S^{2/3} - \tau_c^* \right]^{n_T} \tag{25a,b}$$

Note again the natural occurrence of the parameter χ = A/B_c in the above relation.

SUBMODEL FOR AGING AND BATTERING LAYERS PRODUCING PLUCKABLE CHUNKS

The bedrock is divided in the vertical into three layers; a "battering layer" of thickness D_p at the top, an "aging layer" of thickness L_a below the battering layer, and "intact" bedrock below the "aging layer", as illustrated in Figure 2.

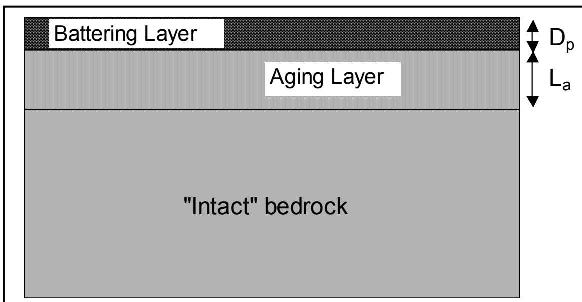


Figure 2

Within the "aging layer" it is assumed that bedrock is gradually breaking up into pluckable chunks of characteristic size D_p over time. Particles in the aging layer are protected from the bedload transport of wear particles by the battering layer above, so the process is assumed to proceed independently of channel transport. While the basic image here is that of rock fracture, the idea can apply to cracking of bedrock in badlands due to repeated wetting and drying, and breakup of partially consolidated clay due to bioturbation as well. Let p_{pa} denote the volume fraction of material in the aging layer that has broken up into pluckable chunks. The following relation is assumed for the time variation of p_{pa};

$$\frac{\partial p_{pa}}{\partial t} = \frac{1}{T_{pa}}(1 - p_{pa}) - \frac{I v_I}{L_a} p_{pa} \tag{26}$$

The first term on the right-hand side of (26) describes an exponential/logistic breakup over time, where T_{pa} is a parameter with the dimension [T] that is proportional to a "half-life" for the aging layer to break up completely into movable chunks. The term (1 – p_p) quantifies the fact that material that has already broken up is no longer available for further breakup. The second term on the right-hand side of (26) describes the rate at which unbroken rock from below is incorporated into the aging layer as the channel incises. The parameter $1/T_{pa}$ can be interpreted as the frequency at which fracturing (or an equivalent mechanism) produces a pluckable chunk within the "aging layer", and is thus probably quantifiable based on field measurements.

Within the "battering layer" of Figure 2, the process of breakup to pluckable sizes is abetted by the collision of wear particles with the bed, a process that has been termed "macroabrasion" (Whipple et al., 2000). Let p_{pb} denote the volume fraction of material in the battering layer that consists of pluckable particles, T_{pb} denote the time constant associated with enhanced breakup due to battering. Within the battering layer, then,

$$\frac{\partial p_{pb}}{\partial t} = \left(\frac{1}{T_{pa}} + \frac{1}{T_{pb}}\right)(1 - p_{pb}) - \frac{I v_I}{D_p}\left(p_{pb} - p_{pa}\right) \tag{27}$$

In the second term on the right-hand side of (27), the term - (p_{pb} – p_{ba}) quantifies the rate at which volume of pluckable particles are lost from the top of the layer (to bedload transport) and gained from the bottom of the layer (from the aging layer) as incision proceeds.

The "aging" and "battering" formulations presented here are somewhat similar to a weathering formulation offered by Howard (1998).

SUBMODEL FOR MACROABRASION

The basic assumption of the submodel for macroabrasion is that the more frequently wear particles strike the surface of the battering layer, higher is the rate of production of pluckable particles due to fracturing or related processes. As noted in the description of the submodel for wear, the number of wear particles that strike the bed per unit time per unit bed area is equal to E_{saltw}/V_w, where V_w denotes the volume of a wear particle. Now let A_p denote some measure of the surface area of a pluckable particle. The number of strikes per unit time on a zone of battering layer surface with area A_p is thus given as $E_{salt}A_p/V_w$. Assuming a linear relation between the number of strikes per unit time on this area and the frequency with this the area fractures into a pluckable particle, then,

$$\frac{1}{T_{pb}} \sim E_{saltw}\frac{A_p}{V_w} \sim E_{saltw}\frac{D_p^2}{D_w^3} \tag{28}$$

Now between (6) and (28) it is seen that

$$E_{saltw}\frac{D_p^2}{D_w^3} = \frac{q_{wear}}{L_{saltw}}\frac{D_p^2}{D_w^3} = \frac{1}{r}\beta_w q_{wear}\frac{D_p^2}{D_w^3} \tag{29}$$

The proportionality (28) is converted to an equality with the aid of (29) and a parameter φ'_{ma}, so that

$$\frac{1}{T_{pb}} = \frac{\varphi'_{ma}}{r}\beta_w q_{wear}\frac{D_p^2}{D_w^3} \equiv \varphi_{ma}\beta_w q_{wear}\frac{D_p^2}{D_w^3} \tag{30}$$

Here φ_{ma} is a dimensionless parameter characterizing macroabrasion.

SUBMODEL FOR INCISION DUE TO PLUCKING

Pluckable chunks are assumed to have size $D_p > D_w$. These are assumed to be entrained into bedload transport by the flow. Let E_p denote the volume rate of entrainment of pluckable chunks into bedload transport per unit bed area per unit time. (This parameter is not to be confused with E_{saltw}, which denotes the volume rate of collosion of wear particles with the bed per unit time per unit bed area per unit time. Once a particle is entrained into bedload, it typically undergoes many saltations before it completes one step length.) Standard relations for entrainment into bedload transport have been available since the time of Einstein. Here the form of a relation due to Tsujimoto (1991) is used;

$$E_p = \sqrt{RgD_p}\,E_p^*$$
$$E_p^* = \gamma_E\left(\frac{\tau_b}{\rho RgD_p} - \tau_{cp}^*\right)^{n_E} \qquad (31a,b)$$

where γ_E is a dimensionless constant, n_E is a dimensionless exponent and $\tau_{cp}{}^*$ is a critical Shields number for entrainment of pluckable grains, which is liable to be somewhat larger than $\tau_c{}^*$ because the grains in question are liable to have funny (block-like) shapes. Fernandez Luque and van Beek (1976) suggest the following values; γ_E = 0.0199 and n_E = 1.5.

The incision rate due to plucking is thus given as

$$v_{Ip} = E_p p_{pb} p_o \qquad (32)$$

According to (32), plucking cannot occur if the bed is covered with alluvium (p_o = 0), nor can it occur if pluckable chunks are not available in the battering layer (p_{pb} = 0). Reducing (32) with (23), (24) and (31) it is found that

$$v_{Ip} = \sqrt{RgD_p}\,\gamma_E\left[\frac{C_f^{1/3} g^{2/3} i^{2/3} \chi^{2/3}}{RgD_p} S^{2/3} - \tau_{cp}^*\right]^{n_E} p_{pb} p_o \qquad (33)$$

Once pluckable grains are removed, they should in principle be added to the wear-effective load q_w. This is not done here simply because the rate of mass production of sediment from direct incision from the bed is liable to be only a tiny fraction of the production of even only wear-effective (coarse) sediment from the hillslopes.

SUMMARY OF THE COMPLETE MODEL

The complete model may be summarized as:

$$(1-\lambda_p)\frac{\partial\eta}{\partial t} = \upsilon - I\left(v_{wI} + v_{Ip}\right)$$
$$v_{Iw} = \beta_w \alpha q\left(1 - \frac{\alpha q}{q_{wearc}}\right)^{n_o}$$
$$q = \frac{m_b + 1}{\chi^{m_b}}\int_0^{\chi}\left(v_{Iw} + v_{Ic}\right)\chi'^{m_b}\,d\chi' \text{ (example relation)}$$
$$q_{wearc} = \sqrt{RgD_w}\,D_w\gamma_T\left[\frac{C_f^{1/3} i^{2/3}\chi^{2/3}}{Rg^{1/3}D_w} S^{2/3} - \tau_c^*\right]^{n_T}$$

$$v_{lp} = E_p p_{pb} \left(1 - \frac{\alpha q}{q_{wearc}}\right)^{n_o}$$

$$E_p = \sqrt{RgD_p}\gamma_E \left(\frac{\tau_b}{\rho R g D_p} - \tau_{cp}^*\right)^{n_E}$$

$$\frac{\partial p_{pb}}{\partial t} = \left(\frac{1}{T_{pa}} + \frac{1}{T_{pb}}\right)(1 - p_{pb}) - \frac{I v_l}{D_p}\left(p_{pb} - p_{pa}\right)$$

$$\frac{\partial p_{pa}}{\partial t} = \frac{1}{T_{pa}}(1 - p_{pa}) - \frac{I v_l}{L_a} p_{pa}$$

$$\frac{1}{T_{pb}} = \varphi_{ma}\beta_w q_{wear} \frac{D_p^2}{D_w^3}$$

The essentially "new" parameters that must be estimated to implement the model are not very numerous, i.e. the wear coefficient β_w [1/L], the aging time constant T_{pa} [T], the thickness of the aging layer L_a [L], the coefficient of macroabrasion φ_{ma} [1], the sizes D_w and D_p (both [L]), the exponent n_o in (9) and the parameter α [1] (for which reasonable estimates are already available, see Sklar, thesis).

A SAMPLE STEADY-STATE SOLUTION FOR UNIFORM UPLIFT

We assume steady-state incision in response to uniform (piston-style) uplift for the fun of it, so that (4a,b) become

$$\left(v_{lw} + v_{lp}\right)I = \upsilon \qquad (34)$$

where υ is assumed to be constant. Note that this assumption precludes the appearance of a stable bedrock-alluvial transition in the reach in question, because v_l must be vanishing downstream of this transition. (But there could be a neat moving boundary if the condition of steady-state incision is relaxed, or a stable boundary if υ is allowed to decrease below zero with increasing x.)

Between (5) and (17) it is found that

$$q_{wear} = \frac{\alpha\upsilon\chi}{I} \qquad (35)$$

Reducing (9) with (25) and (35), it is found that

$$p_o = 1 - \left\{ \frac{\alpha \hat{\upsilon} \hat{\chi}}{\gamma_T \left[\left(\frac{C_f}{R^2} \hat{i}^2 \hat{\chi}^2 \right)^{1/3} S^{2/3} - \tau_c^* \right]^{n_T}} \right\}^{n_o} \tag{36}$$

where the dimensionless uplift rate $\hat{\upsilon}$, dimensionless rainfall rate $\hat{i}$ and dimensionless "equivalent" downstream distance $\hat{\chi}$ are defined as

$$\hat{\upsilon} = \frac{\upsilon}{I\sqrt{RgD_w}} \qquad \hat{i} = \frac{i}{\sqrt{RgD_w}} \qquad \hat{\chi} = \frac{A}{D_w B_c} \tag{37a,b,c}$$

Recall that in order for a reach of a river to be incisional p_o must exceed zero everywhere on it.

In the case of a steady state it is found from (26), (27) and (30) that

$$p_{pa} = \frac{1}{1 + \frac{\hat{\upsilon} I \omega}{\lambda_L}}$$

$$p_{pb} = \frac{\left[1 + \varphi_{ma} \alpha \hat{\beta}_w \hat{\upsilon} \omega I \lambda_p^2 \hat{\chi} + \frac{\hat{\upsilon} I \omega}{\lambda_p} \left(\frac{1}{1 + \frac{\hat{\upsilon} I \omega}{\lambda_L}} \right) \right]}{\left(1 + \varphi_{ma} \alpha \hat{\beta}_w \hat{\upsilon} \omega I \lambda_p^2 \hat{\chi} + \frac{\hat{\upsilon} I \omega}{\lambda_p} \right)}$$

(38a,b)

where

$$\hat{\beta}_w = D_w \beta_w \qquad \omega = \frac{\sqrt{RgD_w} T_{pa}}{D_w} \qquad \lambda_p = \frac{D_p}{D_w} \qquad \lambda_L = \frac{L_a}{D_w} \tag{39a,b,c,d}$$

define four more dimensionless parameters.

Now reducing the relations of the previous section, it is found that

$$\hat{\upsilon} = \left\{ \left(\hat{\beta}_w \alpha \hat{\upsilon} \hat{\chi} \right) + \lambda_p^{1/2} p_{pb} \gamma_E \left[\lambda_p^{-1} \left(\frac{C_f}{R^2} \hat{I}^2 \hat{\chi}^2 \right)^{1/3} S^{2/3} - \tau_{cp}^* \right]^{n_E} \right\} \cdot$$

$$\left\{ 1 - \frac{\alpha \hat{\upsilon} \hat{\chi}}{\gamma_T \left[\left(\frac{C_f}{R^2} \hat{I}^2 \hat{\chi}^2 \right)^{1/3} S^{2/3} - \tau_c^* \right]^{n_T}} \right\}^{n_o} \tag{40}$$

Equation (40) provides a solution in closed but implicit form for channel slope S as a function of dimensionless equivalent downstream distance $\hat{\chi} = A/(B_c D_w)$ and dimensionless uplift rate $\hat{\upsilon}$. The limiting case of pure wear is obtained as $\omega \to \infty$ and $\varphi_{ma} \to 0$. The limiting case of pure plucking is obtained as $\hat{\beta}_w \to 0$.

An explicit solution is easily found for the case of pure wear with $n_o = 1$, in which case

$$S = \frac{R}{C_f^{1/2} \hat{I} \hat{\chi}} \left\{ \left[\frac{\hat{\beta}_w \hat{\upsilon} \alpha^2 \hat{\chi}^2}{\left(\hat{\beta}_w \alpha \hat{\chi} - 1 \right) \gamma_T} \right]^{1/n_T} + \tau_c^* \right\}^{3/2} \tag{41}$$

I was not able to find a general explicit solution for the case of pure plucking. In the limiting case for which $\lambda_p = 1$, $n_o = 1$, $\tau_{cp}^* = \tau_c^*$ and $n_T = n_E$, however, the following solution can be found;

$$S = \frac{R}{C_f^{1/2} \hat{I} \hat{\chi}} \left\{ \left[\frac{\hat{\upsilon}}{\gamma_E p_{pb}} \left(1 + \frac{\gamma_E}{\gamma_T} \alpha \hat{\chi} p_{pb} \right) \right]^{1/n_T} + \tau_c^* \right\}^{3/2} \tag{42}$$

DEVIATION FROM STEADY STATE

In order to characterize the general problem, in which the uplift rate may vary in χ and the long profile of the river may not be at steady state, the following hatted dimensionless parameters are introduced.

$$\eta = D_w \hat{\eta} \qquad t = \frac{1 - \lambda_p}{I \sqrt{\frac{D_w}{R g}}} \hat{t} \qquad v_I = \sqrt{R g D_w}\, \hat{v}_I$$

$$q_{wear} = \sqrt{R g D_w}\, D_w \hat{q}_{wear} \tag{43a,b,c,d}$$

The uplift rate υ is assumed to vary in χ about a characteristic reference value υ_r, so that

$$\upsilon = \upsilon_r f_u(\hat{\chi}) \tag{45}$$

where $f_u(\hat{\chi})$ is a specified dimensionless function. In addition, the parameter υ_r is made dimensionless as

$$\upsilon_r = \sqrt{R g D_w}\,\hat{\upsilon}_r \tag{46}$$

With the above definitions, the time-dependent problem becomes

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} = \hat{\upsilon}_r f_u(\hat{\chi}) - \hat{v}_l \tag{47a}$$

$$\hat{v}_l = \left(\hat{\beta}_w \alpha \hat{q} + \lambda_p^{1/2} E_p^* p_{pb}\right)\left(1 - \frac{\alpha \hat{q}}{q_{wearc}^*}\right)^{n_o} \tag{47b}$$

$$\hat{q} = \frac{m_b + 1}{\hat{\chi}^{m_b}} \int_0^{\hat{\chi}} \hat{v}_l \hat{\chi}'^{m_b} d\hat{\chi}' \tag{47c}$$

$$\frac{\partial p_{pb}}{\partial \hat{t}} = \frac{1}{I\omega}\left(1 + \varphi_{ma} \alpha \hat{\beta}_w \omega \lambda_p^2 I \hat{q}\right)(1 - p_{pb}) - \frac{\hat{v}_l}{\lambda_p}\left(p_{pb} - p_{pa}\right) \tag{47d}$$

$$\frac{\partial p_{pa}}{\partial \hat{t}} = \frac{1}{I\omega}(1 - p_{pa}) - \frac{\hat{v}_l}{\lambda_L} p_{pa} \tag{47e}$$

Solving the above equations will be a bit tricky, but should yield some interesting behavior. To see the tricky part clearly, it is useful to reduce (47b) and (47c) to the single relation

$$\hat{v}_l = \left[\hat{\beta}_w \alpha \left(\frac{m_b + 1}{\hat{\chi}^{m_b}} \int_0^{\hat{\chi}} \hat{v}_l \hat{\chi}'^{m_b} d\hat{\chi}'\right) + \lambda_p^{1/2} E_p^* p_{pb}\right]\left[1 - \frac{\alpha\left(\frac{m_b + 1}{\hat{\chi}^{m_b}} \int_0^{\hat{\chi}} \hat{v}_l \hat{\chi}'^{m_b} d\hat{\chi}'\right)}{q_{wearc}^*}\right]^{n_o} \tag{48}$$

It is seen from the above equation that $\hat{v}_l$ at a point is a function of all values of $\hat{v}_l$ upstream. In order to implement (47a) to compute the change in bed elevation, it is first necessary to determine $\hat{v}_l$ at all points.

To implement this solution, define

$$W = \int_0^{\hat{\chi}} \hat{v}_l \hat{\chi}'^{m_b} d\hat{\chi}' \tag{49a}$$

from which

$$\hat{v}_I = \frac{1}{\hat{\chi}^{m_b}} \frac{dW}{d\hat{\chi}} \tag{49b}$$

Substituting (49b) into (48) yields the differential equation

$$\frac{dW}{d\hat{\chi}} = \hat{\chi}^{m_b} \left[\hat{\beta}_w \alpha \left(\frac{m_b + 1}{\hat{\chi}^{m_b}} W \right) + \lambda_p^{1/2} E_p^* p_{pb} \right] \left[1 - \frac{\alpha \left(\frac{m_b + 1}{\hat{\chi}^{m_b}} W \right)}{q_{wearc}^*} \right]^{n_o} \tag{50}$$

SOME NUMBERS

I have been playing with the following numbers in order to study the steady state problem. Two sample plots are shown. The cases of pure wear with n_o = 1 and pure plucking with λ_p = 1, n_o = 1, τ_{cp}^* = τ_c^* and n_E = n_T are studied.

Numbers used in spreadsheet calculation

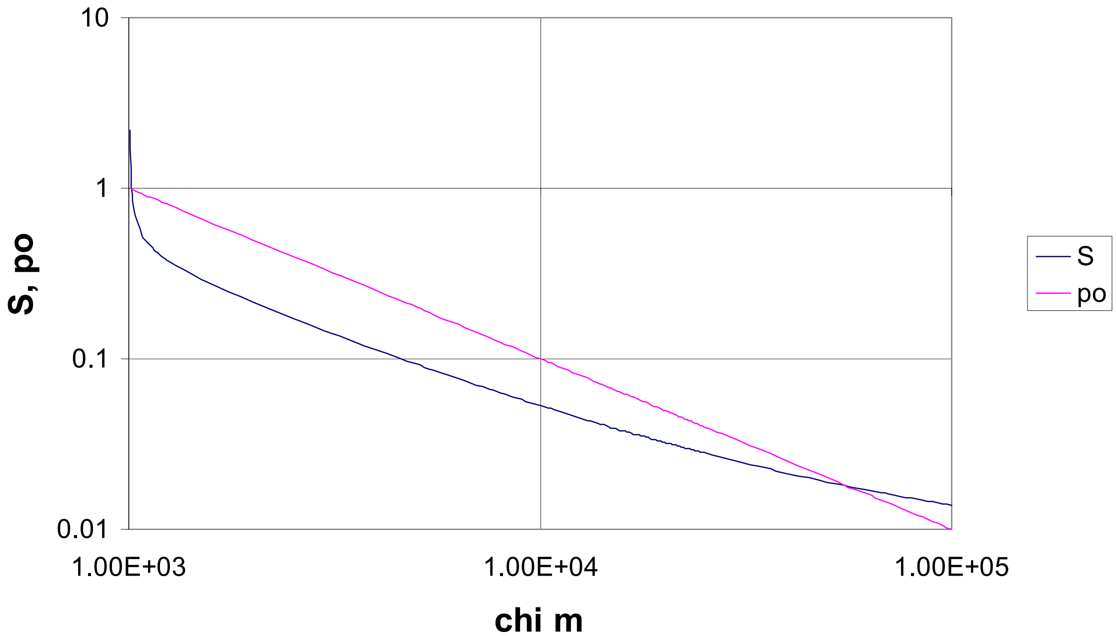
R	1.65
β_w	4.00E-031/m
i	1m/day
υ	0.01m/a
C_f	0.01
T_a	300a
L_a	0.4m
D_w	0.05m
λ_p	1
τ_c^*	0.03
γ_T	5.7
γ_E	0.0199
α	0.25
n_T	1.5
n_E	1.5
I	0.003
φ_{ma}	1.00E-3
n_o	1

These give the dimensionless parameters

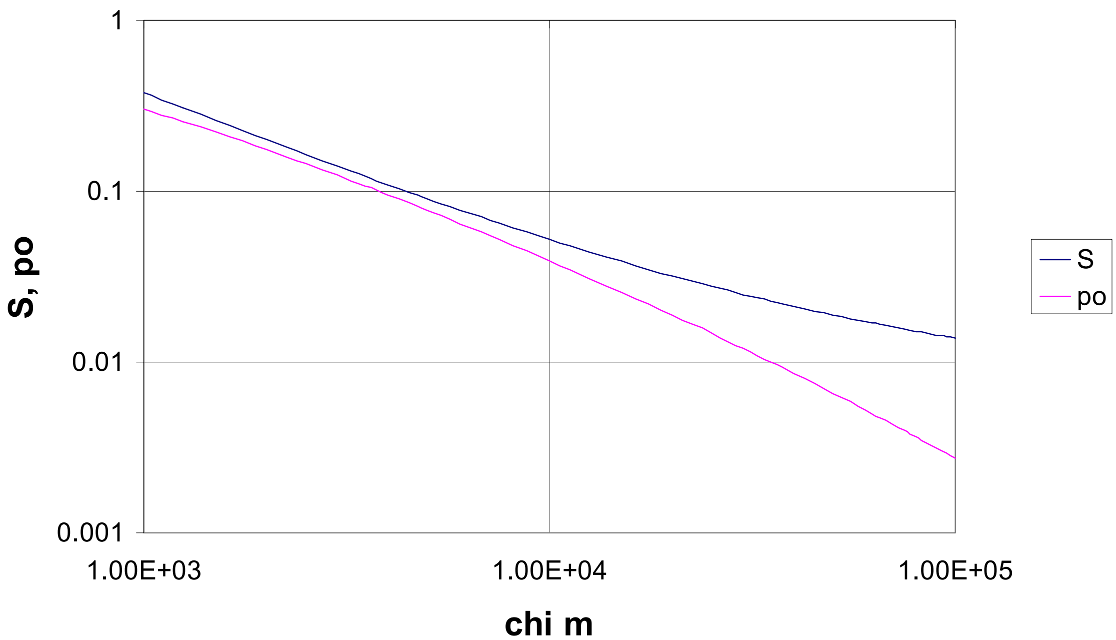
$\hat{\beta}_w$	2.00E-04
$\hat{\upsilon}$	1.17412E-07

$\hat{i}$	1.28654E-05
ω	1.70E+11
p_{pa}	0.117647059
λ_L	8

Wear



Plucking



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