Toward the Holy Grail of River Mechanics:

ON THE ORIGIN OF THE ½ POWER RELATION BETWEEN BANKFULL WIDTH B_b AND BANKFULL DISCHARGE Q_{wb} IN RIVERS

$$B_b \propto Q_{wb}^{1/2}$$

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(with help from:
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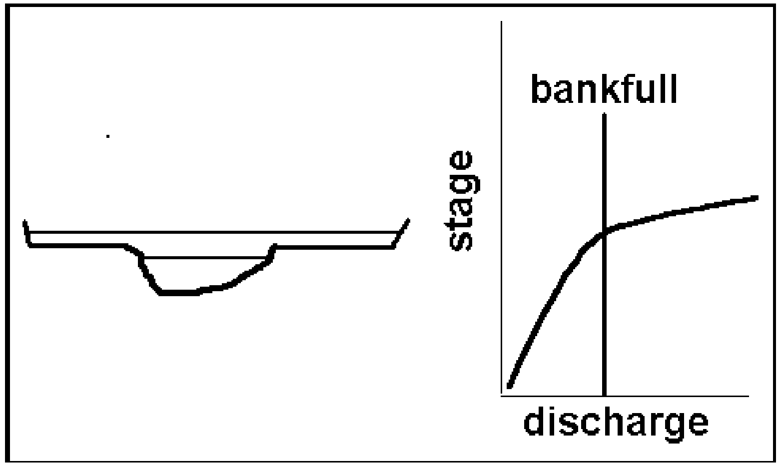
ALLUVIAL RIVERS FORM THEIR OWN CHANNEL AND FLOODPLAIN



KEY PARAMETERS

- Bankfull discharge $\mathbf{Q_{wb}}$ (m^3/s)
- Bankfull depth $\mathbf{H_b}$ (m)
- Bankfull width $\mathbf{B_b}$ (m)
- Bankfull velocity $\mathbf{U_b}$ (m)
- Channel slope **S**
- Characteristic bed material size **D** (m)
- Bed material discharge at bankfull flow $\mathbf{Q_{sb}}$ (m^3/s)
- Sediment submerged specific gravity R ($\cong$ 1.65)

The most universal correlation of river behavior is the half-power law between B_b and $\mathbf{Q_{wb}}$

$$\mathbf{B_b \propto Q_{wb}^{1/2}}$$

$$\mathbf{H_b \propto Q_{wb}^{2/5}}$$

$$\mathbf{U_b \propto Q_{wb}^{1/10};}$$

Leopold and Maddock
USGS Professional Paper 252
1953

(originally formulated for mean discharge)

SOME DIMENSIONLESS PLOTS FOR GRAVEL-BED AND SAND-BED STREAMS

Sand-bed and gravel-bed streams form distinct populations

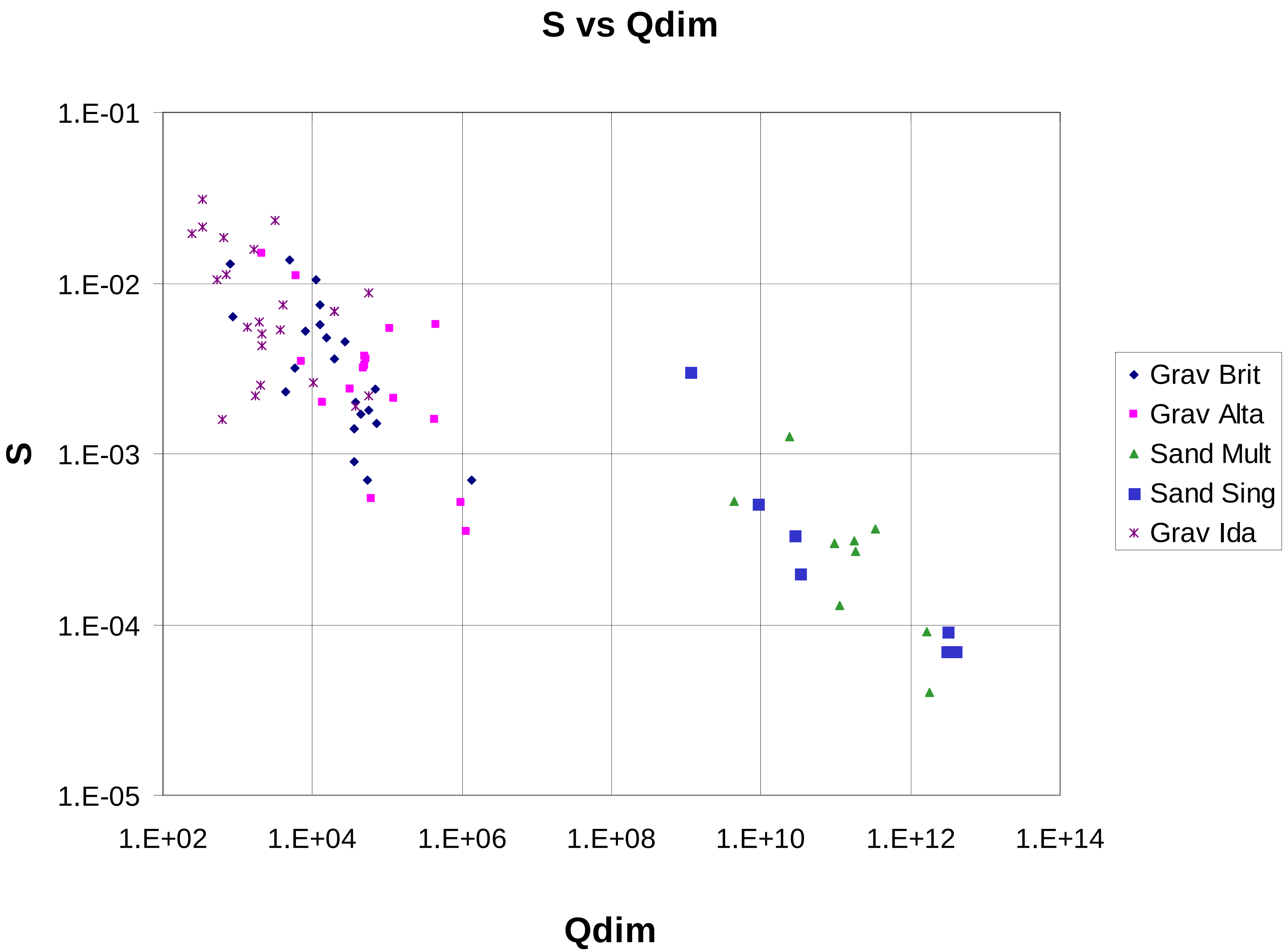
$$Q_{dim} = \frac{Q_{wb}}{\sqrt{gD}\,D^2}$$

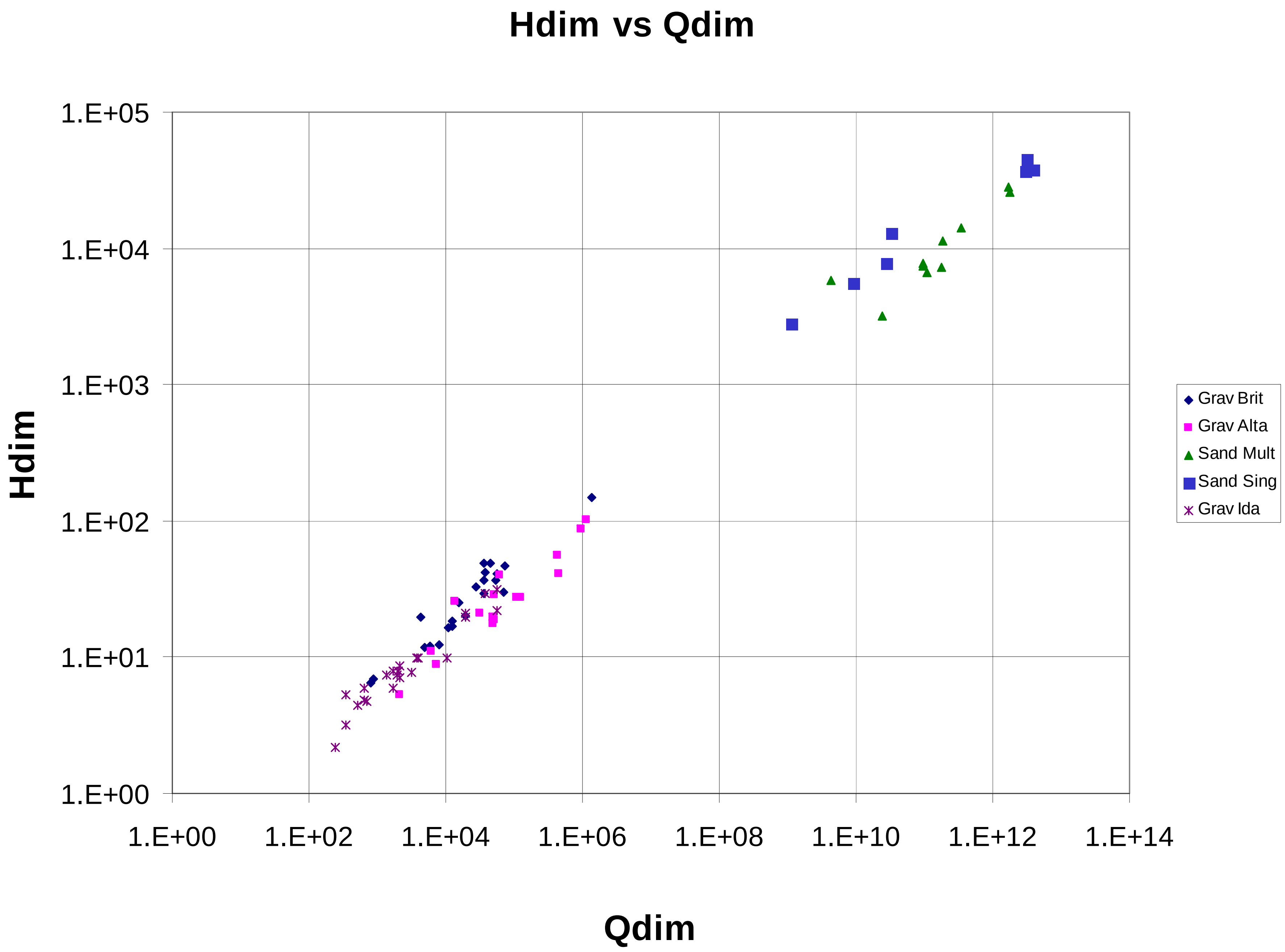
$$B_{dim} = \frac{B_b}{D}$$

$$H_{dim} = \frac{H_b}{D}$$

S

Note validity of ½ power law in width


S vs Qdim
1.E-01
1.E-02
1.E-03
1.E-04
1.E-05
S
1.E+02
1.E+04
1.E+06
1.E+08
1.E+10
1.E+12
1.E+14
Qdim
Grav Brit
Grav Alta
Sand Mult
Sand Sing
Grav Ida



Hdim vs Qdim
Hdim
Qdim
1.E+05
1.E+04
1.E+03
1.E+02
1.E+01
1.E+00
1.E+00
1.E+02
1.E+04
1.E+06
1.E+08
1.E+10
1.E+12
1.E+14
Grav Brit
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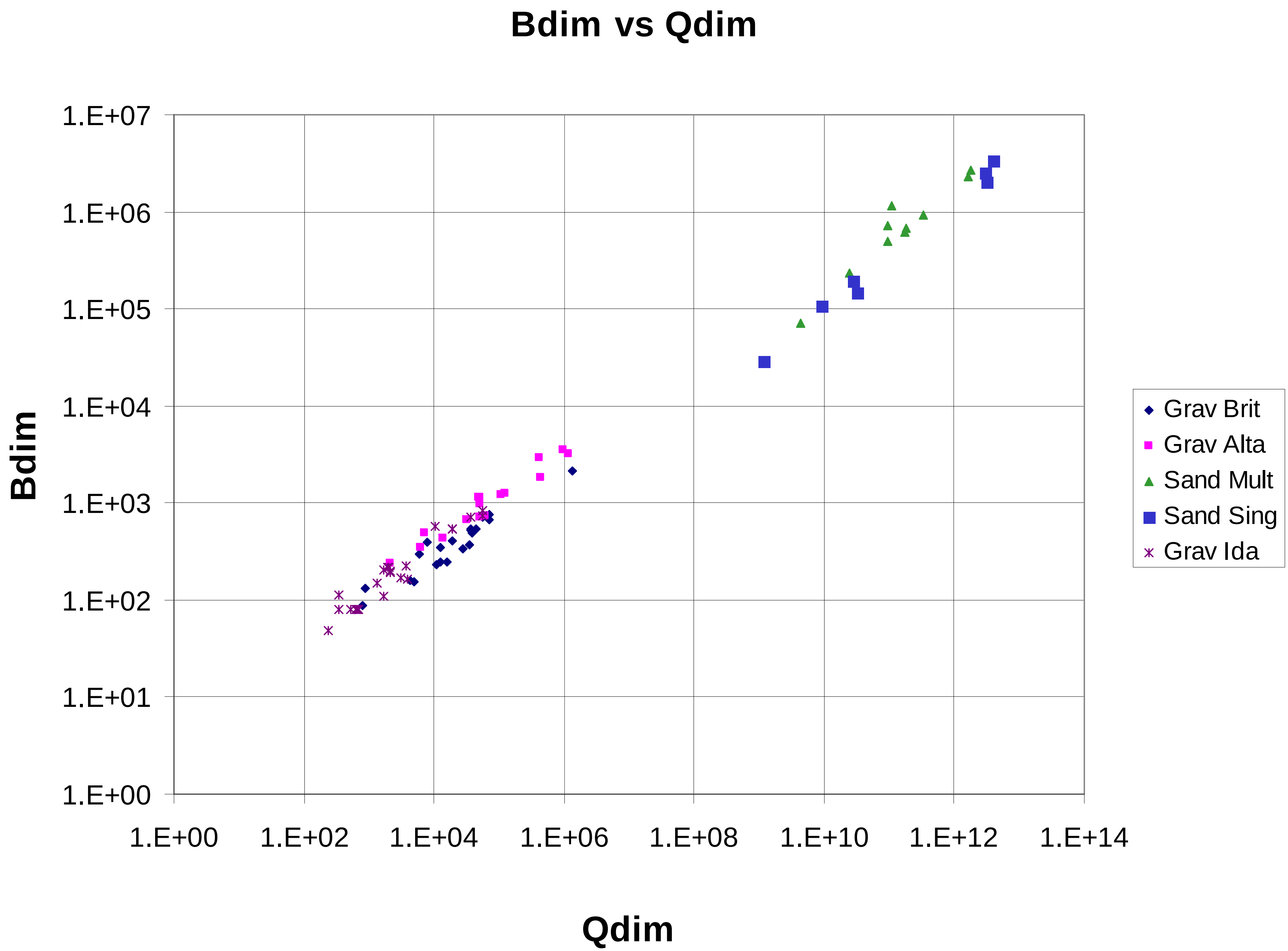


Bdim vs Qdim
Bdim
1.E+07
1.E+06
1.E+05
1.E+04
1.E+03
1.E+02
1.E+01
1.E+00
1.E+00
1.E+02
1.E+04
1.E+06
1.E+08
1.E+10
1.E+12
1.E+14
Qdim
Grav Brit
Grav Alta
Sand Mult
Sand Sing
Grav Ida

Why?

Well, if

$$H_b \propto Q_{wb}^{\chi_h} \; ; \quad B_b \propto Q_{wb}^{\chi_b} \; ; \quad U_b \propto Q_{wb}^{\chi_u}$$

then since $Q_{wb} = H_b B_b U_b$

$$\chi_h + \chi_b + \chi_u = 1$$

Since χ_h, χ_b and χ_u are all expected to be positive, it seems likely that

$$0 < \chi_b < 1$$

But why should $\chi_b = 1/2$?

Is there a deeper meaning?
Does it tell us anything about river mechanics?
Does it tell us anything about basin mechanics?

Some basic flow relations applied to bankfull conditions

CONSERVATION RELATIONS

Water mass conservation:

$$\mathbf{Q_{wb} = H_b B_b U_b} \qquad (1)$$

Sediment mass concentration: where $\mathbf{q_{sb}}$ = the volume sediment transport rate of bed material load per unit width at bankfull flow,

$$\mathbf{Q_{sb} = B_b q_{sb}} \qquad (2)$$

Water momentum conservation (normal flow): where $\mathbf{u_*}$ denotes shear velocity at bankfull flow,

$$\mathbf{u_*^2 = g H_b S} \qquad (3)$$

TWO DIMENSIONLESS PARAMETERS

$$\frac{q_{sb}}{\sqrt{RgD}\,D} = q_{sb}^{*} = \text{Einstein parameter} \qquad (4)$$

$$\frac{u_{*}^{2}}{RgD} = \tau^{*} = \text{Shields stress} \qquad (5)$$

both at bankfull conditions

INTERNAL RELATIONS

Flow resistance (generalized Manning-Strickler)

$$\frac{U_b}{u_*} = \alpha_r \left(\frac{H_b}{D}\right)^{p} \qquad (6)$$

Chezy: p = 0; Strickler; p = 1/6

Bed material transport

$$q^{*} = \alpha_s (\tau^{*} - \tau_c^{*})^{n} \qquad (7)$$

where τ_c^* = critical Shields stress for sediment motion (n ~ 1.5 to 2.5)
Note α_r, α_s are dimensionless

CHANNEL FORMING RELATION

Recalliing that τ^* refers to bankfull flow here,

$$\tau^* = \tau^*_{ch} \qquad (8)$$

where

τ^*_{ch} ~ 0.050 for gravel-bed streams

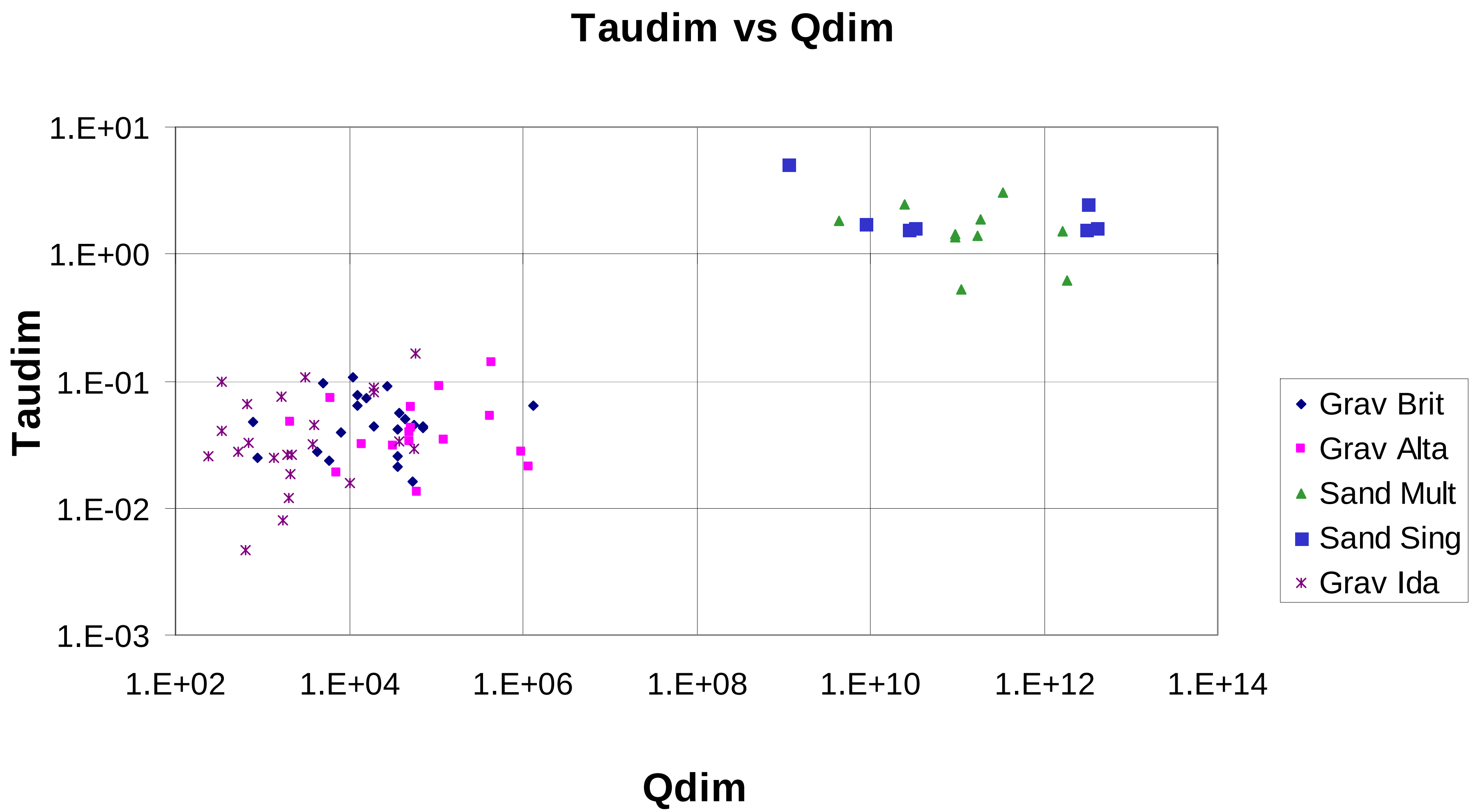
τ^*_{ch} ~ 1.8 for sand-bed streams

Between momentum conservation (3) and the definition of the Shields stress (5), then,

$$\frac{H_b S}{R D} = \tau^*_{ch} \qquad (9)$$

That is, bankfull depth is determined as a function of S, D etc.

Justification: Parker 1978 a,b *JFM*



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THE ABOVE FORMULATION REDUCES TO THE FOLLOWING RELATIONS:

$$\frac{\mathbf{B_b}}{\mathbf{D}} = \mathbf{b_b}\,\mathbf{S}^{\mathbf{1+p}}\left(\frac{\mathbf{Q_{wb}}}{\sqrt{\mathbf{gD}}\,\mathbf{D}^2}\right) \quad (10)$$

$$\frac{\mathbf{H_b}}{\mathbf{D}} = \mathbf{b_h}\,\mathbf{S}^{-1} \quad (11)$$

$$\frac{\mathbf{Q_{sb}}}{\mathbf{Q_{wb}}} = \mathbf{b_s}\,\mathbf{S}^{\mathbf{1+p}} \quad (12)$$

where the dimensionless coefficients are

$$\mathbf{b_b} = (R\tau^*_{\mathbf{ch}})^{-(3+2p)/2}\,\alpha_r^{-1} \quad (13)$$

$$\mathbf{b_h} = R\tau^*_{\mathbf{ch}} \quad (14)$$

$$\mathbf{b_s} = R^{1/2}\,\alpha_s\,(\tau^*_{\mathbf{ch}} - \tau^*_{\mathbf{c}})^{\mathbf{n}}\,\mathbf{b_b} \quad (15)$$

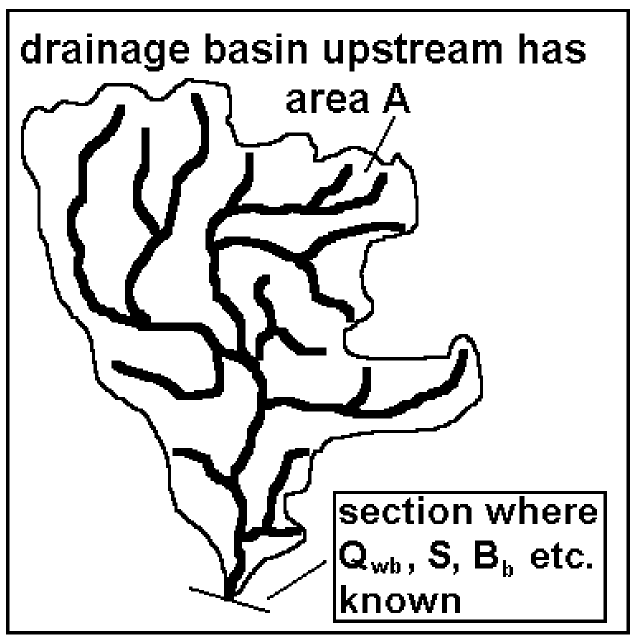
These THREE relations constrain the SIX parameters $\mathbf{Q_{wb}}$, $\mathbf{Q_{sb}}$, $\mathbf{S}$, $\mathbf{B_b}$, $\mathbf{H_b}$, $\mathbf{D}$

(10) gives $\mathbf{B_b} = \mathrm{fn}(\mathbf{Q_{wb}}, \mathbf{S}, \mathbf{D})$
Cannot reduce further,
thus canot get $\mathbf{B_b} \propto (\mathbf{Q_{wb}})^{1/2}$

THE EPIPHANY

Perhaps the missing information concerning the ½ power law is in how drainage basins organize themselves to deliver water and sediment!

Let A denote the area of the drainage basin upstream of a section where $\mathbf{Q_{wb}}$**, S¸D** etc. are known. For simplicity, the mean annual rainfall $\mathbf{i_w}$ and grain size **D** are taken to be constant over basin.



Simple relation for water conservation over drainage basin:

$$\mathbf{Q}_{wb} = \lambda \mathbf{i}_w \mathbf{A} \tag{16}$$

where λ is a conversion factor from mean to bankfull

POSTULATE THE FOLLOWING!

Dimensionless structure functions for drainage basin formation:

$$\frac{\mathbf{Q}_{wb}}{\mathbf{g}^{1/2}\mathbf{A}^{5/4}} = \mathbf{c}_w \mathbf{S}^{\beta_w} \tag{17}$$

$$\frac{\mathbf{Q}_{sb}}{\mathbf{g}^{1/2}\mathbf{A}^{5/4}} = \mathbf{c}_s \mathbf{S}^{\beta_s} \tag{18}$$

Hope: while $\mathbf{c}_w$ *and* $\mathbf{c}_s$ *might be functions of rate of regolith formation, rock type, hydrologic regime etc.* β_w and β_s *might be universal*

Non-dimensionalization based on effective rainfall rate $\lambda \mathbf{i_w}$ and gravitational acceleration $\mathbf{g}$:

$$\tilde{Q}_w = \frac{g^2 Q_{bw}}{(\lambda i_w)^5}; \quad \tilde{Q}_s = \frac{g^2 Q_{bs}}{(\lambda i_w)^5}; \quad \tilde{A} = \frac{g^2 A}{(\lambda i_w)^4}$$

$$\tilde{B} = \frac{g B_b}{(\lambda i_w)^2}; \quad \tilde{H} = \frac{g H_b}{(\lambda i_w)^2}; \quad \tilde{D} = \frac{g D}{(\lambda i_w)^2}$$

BASIN RELATIONS:

(16) becomes

$$\tilde{Q}_w = \tilde{A} \qquad (19)$$

(17) and (18) become

$$\tilde{Q}_w = c_w \tilde{A}^{5/4} S^{\beta_w} \qquad (20)$$

$$\tilde{Q}_s = c_s \tilde{A}^{5/4} S^{\beta_s} \qquad (21)$$

Between (19) and (20) it is found that

$$\mathbf{S} = \mathbf{c}_{\mathbf{w}}^{-1/\beta_w} \, \tilde{\mathbf{A}}^{-1/(4\beta_w)} = \mathbf{c}_{\mathbf{w}}^{-1/\beta_w} \, \tilde{\mathbf{Q}}_{\mathbf{w}}^{-1/(4\beta_w)} \quad (22)$$

CHANNEL RELATIONS:

(10), (11) and (12) reduce with (19) to

$$\tilde{\mathbf{B}} = \mathbf{b}_{\mathbf{b}} \, \tilde{\mathbf{Q}}_{\mathbf{w}} \, \tilde{\mathbf{D}}^{-3/2} \, \mathbf{S}^{1+p} \quad (23)$$

$$\tilde{\mathbf{H}} = \mathbf{b}_{\mathbf{h}} \, \tilde{\mathbf{D}} \, \mathbf{S}^{-1} \quad (24)$$

$$\tilde{\mathbf{Q}}_{\mathbf{s}} = \mathbf{b}_{\mathbf{s}} \tilde{\mathbf{Q}}_{\mathbf{w}} \, \mathbf{S}^{1+p} \quad (25)$$

Now use (22) *to eliminate* **S** *from the above three relations*!!!

Thus

$$\tilde{\mathbf{B}} = \mathbf{d_b}\, \tilde{\mathbf{D}}^{-3/2}\, \tilde{\mathbf{Q}}_{\mathbf{w}}^{\chi_{\mathbf{b}}} \qquad (23)$$

$$\tilde{\mathbf{H}} = \mathbf{d_h}\, \tilde{\mathbf{D}}\, \tilde{\mathbf{Q}}_{\mathbf{w}}^{\chi_{\mathbf{h}}} \qquad (24)$$

$$\tilde{\mathbf{Q}}_{\mathbf{s}} = \mathbf{d_s}\, \tilde{\mathbf{Q}}_{\mathbf{w}}^{\chi_{\mathbf{s}}} \qquad (25)$$

and rewriting (22),

$$\mathbf{S} = \mathbf{d_{sl}}\, \tilde{\mathbf{Q}}_{\mathbf{w}}^{\chi_{\mathbf{sl}}} \qquad (26)$$

where coefficients are

$$\mathbf{d_b} = \mathbf{b_b c_w^{-(1+p)/\beta_w}}\ ; \quad \mathbf{d_h} = \mathbf{b_h c_w^{1/\beta_w}}$$

$$\mathbf{d_s} = \mathbf{b_s c_w^{-(1+p)/\beta_w}}\ ; \quad \mathbf{d_{sl}} = \mathbf{c_w^{-1/(4\beta_w)}}$$

(27a – d)

and exponents are

$$\chi_b = 1 - \frac{1+p}{4\beta_w}; \quad \chi_h = \frac{1}{4\beta_w}$$

$$\chi_s = \chi_b; \quad \chi_{sl} = -\chi_h$$

(28a – d)

Finally, between (21) and (25),

$$c_s = b_s c_w; \quad \beta_s = \beta_w + 1 + p \qquad (29a – b)$$

WE ARE FINALLY IN A POSITION TO START ANSWERING QUESTIONS!

Suppose we accept the ½ power law as an empirical FACT.

HOW MUST THE DRAINAGE BASIN THEN ORGANIZE ITSELF SO AS TO SATISFY THIS VALUE???

Assumption 1: $\mathbf{p} = 1/6$
(Manning-Strickler resistance)

Assumption 2: $\chi_{\mathbf{b}} = 1/2$

It then follows from (28a) that

$\beta_{\mathbf{w}} = 7/12$

and thus

$\beta_{\mathbf{s}} = 7/4$
$\chi_{\mathbf{h}} = 3/7$
$\chi_{\mathbf{s}} = 1/2$
$\chi_{\mathbf{sl}} = -3/7$

Summarizing:

_IF_ DRAINAGE BASINS ORGANIZE THEMSELVES SUCH THAT

$$\tilde{Q}_w = c_w \tilde{A}^{5/4} S^{7/12}$$

AND

$$\tilde{Q}_w = \tilde{A}$$

THEN THE FOLLOWING RELATIONS MUST HOLD:

$$\tilde{B} = d_b \tilde{D}^{-3/2} \tilde{Q}_w^{1/2}$$

$$\tilde{H} = d_h \tilde{D} \tilde{Q}_w^{3/7}$$

$$\tilde{Q}_s = d_s \tilde{Q}_w^{1/2}$$

$$S = d_{sl} \tilde{Q}_w^{-3/7}$$

$$\tilde{Q}_s = c_s \tilde{A}^{5/4} S^{7/4}$$

IS THIS REALLY TRUE?

I DON'T KNOW!!!

The exponents of 3/7 and – 3/7 for bankfull depth and slope are, however, close to the observed ones!!

I bet that "mature" basins that have achieved some state of self-similarity do indeed evolve toward a structure close to the one above.

We need some data!!

DEM folks: please help!

HOW MUCH WOULD YOU PAY TO HAVE THIS THEORY AS YOUR VERY OWN?

WAIT, DON'T ANSWER YET, YOU ALSO GET THIS RESULT ON DENUDATION RATE:

Let v_d denote a denudation rate of the basin in, say, m/s (mm/year)

We can probably scale this as

$$\mathbf{v_d} = \lambda_\mathbf{d} \frac{\mathbf{Q_{sb}}}{\mathbf{A}}$$

This plus the other scalings results in

$$\mathbf{v_d} \propto \mathbf{Q_{wb}^{-1/2}} \propto \mathbf{A^{-1/2}} \propto \mathbf{S^{7/6}}$$

That is, the larger the basin is the lower the denudation rate should be

NOW HOW MUCH WOULD YOU PAY FOR THIS THEORY?