A Kolmogorov-type Scaling for the Fine Structure of Drainage Basins

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Kolmogoroff scaling in turbulent flows: *Energy Cascade*

- Turbulent energy is produced in eddies scaling with the size of the "box" (e.g. river depth).
- The nonlinear terms in the equations of momentum balance grind larger eddies into ever smaller eddies.
- The grinding continues until the eddies are so small that the turbulent energy can be effectively dissipated into heat.
- There are no smaller eddies.
Balance of energy in a steady turbulent flow

\[
\frac{\partial}{\partial t} \text{(mean kinetic energy of turbulence)} = \\
\text{production rate} - \text{dissipation rate} + \text{spatial transfer rate}
\]

Let

\begin{align*}
    u & = \text{characteristic turbulent velocity (L/T)} \\
    L & = \text{size of the container (e.g. river depth) (L)} \\
    \nu & = \text{viscosity of flow (L}^2/\text{T)} \\
    P & = \text{production rate/mass of turbulent energy (L}^2/\text{T}^3) \\
    \varepsilon & = \text{dissipation rate/mass of turbulent energy (L}^2/\text{T}^3) \\
    u_k & = \text{Kolmogoroff velocity scale (L/T)} \\
    \eta_k & = \text{Kolmogoroff length scale}
\end{align*}

\[
P = -u_i'u_j \frac{\partial u_i}{\partial x_j} \sim \frac{u^3}{L} \\
\varepsilon = \nu \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \sim \nu \frac{u_k^2}{\eta_k^2}
\]
Make Kolmogoroff scales from the viscosity $\nu$ (L$^2$/T) and the dissipation rate $\varepsilon$ (L$^2$/T$^3$)

$$
\eta_k = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4}
$$

$$
\frac{u_k}{\nu} = 1
$$

Recalling that

$$
P \approx \varepsilon \quad \varepsilon \sim \frac{u^3}{L}
$$

it is found that

$$
\frac{\eta_k}{L} \sim \left( \frac{\nu}{uL} \right)^{3/4}
$$

That is, the higher the Reynolds number of the flow the finer is the turbulence.
Figure 1.6. Turbulent jets at different Reynolds numbers: (a) relatively low Reynolds number, (b) relatively high Reynolds number (adapted from a film sequence by R. W. Stewart, 1969). The shading pattern used closely resembles the small-scale structure of turbulence seen in shadowgraph pictures.
Does the idea of a Kolmogoroff scale have any application to drainage basins?
How dense does a drainage network have to be in order to "cover" the catchment?
Well, how dense?
Consider a (statistically) steady state landscape undergoing uplift at constant rate $v_u$ (L/T).

The channels are undergoing incision at rate $v_i$ (L/T).

The hillslopes contain a well-developed regolith which moves downslope with a kinematic diffusivity $k$ (L$^2$/T)

**HYPOTHESIS 1:** Channel incision creates elevation fluctuations.

**HYPOTHESIS 2:** Hillslope diffusion obliterates elevation fluctuations.

**HYPOTHESIS 3:** The drainage network must be just sufficiently fine so that rate of creation of elevation fluctuations balances rate of obliteration.
Channel incision in the near-absence of hillslope diffusion: the SLOT CANYON: creates amplitude fluctuation
Hillslope diffusion with weak channel incision: MELTED ICE-CREAM MORPHOLOGY: destroys amplitude fluctuation

Image courtesy Bill Dietrich
Some scales

$L_u = \text{length of headwater tributary}$

$A_u = \alpha_u L_u^2 = \text{area of headwater catchment: } \alpha_u \sim o(1)$

$S_h = \text{characteristic slope of hillslope in headwater catchment}$
Some parameters

\( \eta \) = bed elevation
\( t \) = time
\((x_1, x_2)\) = spatial coordinate
\( \zeta \) = incision rate
\((q_1, q_2)\) = vector of volume hillslope transport/width
\( k \) = hillslope diffusivity

\[ \vec{q} = -k \vec{\nabla} \eta \]

Exner equation of sediment balance (neglecting porosity)

\[ \frac{\partial \eta}{\partial t} = -\zeta + k \nabla^2 \eta + \nu_u \]
Now for simplicity we assume that

- All the hillslope transport occurs on the hillslope and
- All the incision occurs in the channel

Integrate Exner on hillslope

\[
\int \int \frac{\partial \eta}{\partial t} \, dA = v_u A_u - \int \int \nabla \cdot \vec{q} \, dA = v_u A_u - \oint q_n \, ds
\]

gain due to uplift
loss due to transport from hillslope to channel

\[ q_n = -k \frac{\partial \eta}{\partial n} \sim kS_h \]

Here \( n \) is normal to red perimeter of path integral:
Continue integration on hillslope: path integral is around red line

\[ q_n = -k \frac{\partial \eta}{\partial n} \sim kS_h \]

\[ \int q_n \, ds \sim 2L_u q_n \sim 2L_u kS_h \]

The following scale relation results:

\[ v_u A_u = \alpha_u v_u L_u^2 = 2\alpha_h L_u kS_h \]

where \( \alpha_h \) is an \( o(1) \) parameter

That is, the rate of hillslope denudation must just balance with rate of uplift
Incision rate: we assume

\[ \zeta = \begin{cases} v_l \text{ in channel} \\ 0 \text{ on hillslope} \end{cases} \]

Thus within the channel of the headwater tributary

Channel incision must just keep pace with uplift

\[ \frac{\partial \eta}{\partial t} = -v_l + v_u \]

or thus

\[ v_l = v_u \]
Mean and fluctuating bed elevation

\[ \eta' = \eta - \bar{\eta} \]
Video clip

Hasbargen and Paola
Equation of evolution of amplitude of elevation fluctuation

Decompose into mean and fluctuating parts:

$$\eta = \bar{\eta} + \eta' \quad \zeta = \bar{\zeta} + \zeta'$$

where the overbar denotes an average over an appropriate spatial window and the prime denotes a fluctuation about that average.

Local Exner:

$$\frac{\partial \bar{\eta}}{\partial t} = -\bar{\zeta} + k \nabla^2 \bar{\eta} + v_u$$
Local Exner:

\[ \frac{\partial \eta}{\partial t} = -\zeta + k\nabla^2 \eta + v_u \]

Spatially averaged Exner:

\[ \frac{\partial \bar{\eta}}{\partial t} = -\bar{\zeta} + k\nabla^2 \bar{\eta} + v_u \]

Multiply by \( \bar{\eta} \) and reduce:

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \eta^2 \right) = -\bar{\zeta} \bar{\eta} + k \bar{\eta} \nabla^2 \bar{\eta} + v_u \bar{\eta} = -\bar{\zeta} \bar{\eta} + k \bar{\nabla} \cdot (\bar{\eta} \bar{\nabla} \bar{\eta}) - k (\bar{\nabla} \bar{\eta}) \cdot (\bar{\nabla} \cdot \bar{\eta}) + v_u \bar{\eta} \]

Multiply local Exner by \( \eta \), average and reduce:

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \eta^2 + \frac{1}{2} \eta'^2 \right) = -\bar{\zeta} \bar{\eta} - \bar{\zeta}' \bar{\eta}' + k \bar{\nabla} \cdot (\bar{\eta} \bar{\nabla} \bar{\eta}) - k (\bar{\nabla} \bar{\eta}) \cdot (\bar{\nabla} \cdot \bar{\eta}) + v_u \bar{\eta} \]

\[ + k \bar{\nabla} \cdot (\bar{\eta}' \bar{\nabla} \bar{\eta}') - k (\bar{\nabla} \bar{\eta}') \cdot (\bar{\nabla} \bar{\eta}') \]
Subtract B from A to get equation of evolution of amplitude of elevation fluctuation:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \eta'^2 \right) = -\zeta' \eta' + k \nabla \cdot \left( \eta' \nabla \eta' \right) - k (\nabla \eta') \cdot (\nabla \eta')
\]

I. Time rate of change of amplitude of elevation fluctuation
II. Rate of creation of amplitude fluctuation due to incision
III. Rate of transport of amplitude fluctuation due to diffusion
IV. Rate of destruction of amplitude fluctuation by hillslope diffusion
Steady state: approximate balance between creation and destruction of amplitude fluctuation

\[ -\zeta'\eta' \approx k\left(\nabla\eta'\right) \cdot \left(\nabla\eta'\right) \]

Now if most of the destruction is biased toward the finest scale of the drainage basin, i.e. the headwater catchment, then

\[ k\left(\nabla\eta'\right) \cdot \left(\nabla\eta'\right) = \alpha_dKS_h^2 \]

where \( \alpha_d \) is an o(1) constant
Approximate form for the rate of creation of amplitude fluctuation by incision

\[ \eta_c(t) - \eta_c(t + \Delta t) \]

\[ v_I \Delta t \]

In the headwater catchment, approximate the channel as incising into a plain with constant elevation \( \eta_p \). Channel elevation \( \eta_c \) decreases with speed \( v_I \).

Thus:

\[ \bar{\eta} = \frac{\eta_p A_u + \eta_c B L_u}{A_u + B L_u} \]

where B denotes channel width
Now
\[
\frac{1}{2} (\eta')^2 = \begin{cases} 
\frac{1}{2} (\eta_p - \bar{\eta})^2 & \text{on plain} \\
\frac{1}{2} (\eta_c - \bar{\eta})^2 & \text{in channel} 
\end{cases}
\]

Since
\[
\dot{\eta}_c = -V_l = -V_u
\]

it follows that
\[
-\eta' \zeta' = \frac{\partial}{\partial t} \frac{1}{2} (\eta')^2 \bigg|_{\text{incision}} = \frac{\partial}{\partial t} \frac{1}{2} (\eta')^2 \bigg|_{\text{plain}} \frac{A_u + \partial}{\partial t} \frac{1}{2} (\eta')^2 \bigg|_{\text{channel}} + \frac{B}{L_u} \frac{B}{L_u}
\]

\[
(\bar{\eta} - \eta_c) v_u \frac{B}{L_u} \frac{B}{L_u}
\]
Now scaling

$$\bar{\eta} - \eta_c = \alpha_c H$$

where $H$ is an "effective" flow depth and $\alpha_c$ is an $o(1)$ constant.

Thus the rate of creation of fluctuation amplitude by incision is expressed as

$$-\eta' \zeta' = \alpha_c H \nu_u \frac{B L_u}{A_u}$$

Balancing this against the rate of destruction of fluctuation amplitude,

$$\alpha_c H \nu_u \frac{B L_u}{A_u} = \alpha_d k S_h^2$$
Scale relations for headwater catchment:

Geometric scaling:

\[ A_u = \alpha_u L_u^2 \]

Rate of hillslope denudation must balance with rate of uplift:

\[ \alpha_u v_u L_u^2 = 2\alpha_h L_u k S_h \]

Rates of creation and destruction of elevation fluctuation amplitude must balance:

\[ \alpha_c H v_u \frac{B L_u}{A_u} = \alpha_d k S_h^2 \]

**FROM THESE WE OBTAIN**

\[ \frac{L_u}{A_e^{1/2}} = \alpha_1 \left( \frac{k}{v_u A_e^{1/2}} \right)^{1/3} \]

\[ S_h = \alpha_2 \left( \frac{v_u A_e^{1/2}}{k} \right)^{2/3} \]

where \( A_e = BH = \) effective channel area and

\[ \alpha_1 = \frac{1}{\alpha_u} \left( 4 \frac{\alpha_c}{\alpha_d} \alpha_h^2 \right)^{1/3} \]

\[ \alpha_2 = \frac{1}{2} \left( 4 \frac{\alpha_c}{\alpha_d \alpha_h} \right)^{1/3} \]
Let $A_T = \text{the total area of the drainage basin}$ and $L_T = (A_T)^{1/2}$ denote a length scale for the total basin. It then follows that

$$\frac{L_u}{L_T} = \frac{1}{\alpha_u} \left( 4 \frac{\alpha_c}{\alpha_d} \alpha_h^2 \right)^{1/3} \left( \frac{k}{v_u A_e^{1/2}} \right)^{1/3} \left( \frac{A_e^{1/2}}{L_T} \right)$$

Now suppose that $A_e$ is set. Then the ratio $L_u/L_T$ becomes smaller (the drainage basin becomes more intricate) as

a) the size of the basin $L$ increases,

b) the rate of uplift $v_u$ increases and

c) the hillslope diffusivity $k$ decreases.

In addition, $S_h$ becomes larger (headwater hillslopes become steeper) as $v_u$ increases and $k$ decreases.
"Kolmogoroff" Scaling for Drainage Basins

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>10.86</td>
<td>channel incision parameter</td>
</tr>
<tr>
<td>$\alpha_2$</td>
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<td>amplitude dissipation parameter</td>
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<td>$\alpha_c$</td>
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<td>$\zeta$</td>
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Sample calculation (numbers for $k$, $v_u$ courtesy Bill Dietrich, Oregon Pacific Coast Range)

\[
\begin{align*}
\frac{L_u}{A_{e}^{1/2}} &= \alpha_1 \left( \frac{k}{v_u A_{e}^{1/2}} \right)^{1/3} \\
S_h &= \alpha_2 \left( \frac{v_u A_{e}^{1/2}}{k} \right)^{2/3}
\end{align*}
\]

\[
\begin{align*}
\alpha_1 &= \frac{1}{\alpha_u} \left( 4 \frac{\alpha_c}{\alpha_d \alpha_h} \right)^{1/3} \\
\alpha_2 &= \frac{1}{2} \left( 4 \frac{\alpha_c}{\alpha_d \alpha_h} \right)^{1/3}
\end{align*}
\]
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<td>H</td>
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<tr>
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\[
\frac{L_u}{A_e^{1/2}} = \alpha_1 \left( \frac{k}{v_u A_e^{1/2}} \right)^{1/3}
\]

\[
S_h = \alpha_2 \left( \frac{v_u A_e^{1/2}}{k} \right)^{2/3}
\]

\[
\alpha_1 = \frac{1}{\alpha_u} \left( 4 \frac{\alpha_c}{\alpha_d \alpha_h} \right)^{1/3}
\]

\[
\alpha_2 = \frac{1}{2} \left( 4 \frac{\alpha_c}{\alpha_d \alpha_h} \right)^{1/3}
\]
THE END!!
Or maybe not!

But wait! It's not over yet! Can we explain how submarine drainage basins form?

Trinity Canyon and associated drainage network, Eel Margin, Northern California
And what about the fine scale of tidal drainage networks?

Barnstable Salt Marsh, Cape Cod, Massachusetts

*Image courtesy Tao Sun, Sergio Fagherazzi and David Furbish*
Sample calculation

Consider two rivers:
   a "prototype" with mean velocity $U = 4 \text{ m/s}$ and depth $H = 2 \text{ m}$, and
   a "model" with mean velocity $U = 1 \text{ m/s}$ and depth $H = 0.125 \text{ m}$.

One is a perfect Froude model of the other. Note $L \sim H$, $u \sim U/10$, $\nu = 1 \times 10^{-6} \text{ m}^2/\text{s}$.

Prototype $\eta_k \sim 0.075 \text{ mm}$
Eddies range from $\sim 2 \text{ m}$ to $0.075 \text{ mm}$

Model $\eta_k \sim 0.11 \text{ mm}$
Eddies range from $\sim 0.125 \text{ m}$ to $0.11 \text{ mm}$

Scale model up to prototype:
Eddies range from $\sim 2 \text{ m}$ to $1.76 \text{ mm}$